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Continuity of Type-I Intermittency from a Measure-Theoretical Point of View

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We study a simple dynamical system which displays a so-called "type-I intermittency" bifurcation. We determine the Bowen–Ruelle measure μ_ε and prove that the expectation $\mu_\varepsilon(g)$ of any continuous function g and the Kolmogoroff–Sinai entropy $h(\mu_\varepsilon)$ are continuous functions of the bifurcation parameter ε. Therefore the transition is continuous from a measure-theoretical point of view. Those results could be generalized to any similar dynamical system.

KEY WORDS: Onset of turbulence; dynamical systems; bifurcations; invariant measures.

1. INTRODUCTION

It is now well known that there are many different ways to turbulence in fluid systems. Among them intermittency stands out as one of the most interesting since it displays a seemingly gradual transition to turbulence. Such a transition has been observed in experiments on the Rayleigh–Bénard thermogravitational instability in confined geometry by P. Bergé and M. Dubois (experiment on silicone oil with high values of the Prandtl number[(4)]) and A. Libchaber and J. Maurer (experiment on liquid helium 4 with small values of the Prandtl number[(5)]) P. Bergé and M. Dubois observe the following sequence of bifurcations when the Rayleigh number Ra is increased:

(i) First stationary convection sets in for $\mathrm{Ra} = \mathrm{Ra}_1$.

(ii) Then a time-dependent monoperiodic regime takes its place ($\mathrm{Ra} = \mathrm{Ra}_2$).

(iii) For still higher values of the Rayleigh number ($\mathrm{Ra} \geqslant \mathrm{Ra}_3$), the

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system undergoes a new transition (intermittency). The whole structure is randomly disrupted by bursts of turbulence; the periods of quiet monoperiodic regime become shorter and shorter as the Rayleigh number is increased up from Ra until convection becomes fully turbulent.

In Libchaber and Maurer's experiment, the bifurcation scheme is slightly different:

(i) Stationary convection sets in for $\mathrm{Ra} = \mathrm{Ra}_1$.

(ii) For $\mathrm{Ra} = \mathrm{Ra}_2$ convection becomes monoperiodic with a low-frequency f_1 probably associated with a varicose instability.

(iii) For $\mathrm{Ra} = \mathrm{Ra}_3$ a new frequency f_2 associated with transverse oscillations of the rolls appears in the spectrum. f_1 and f_2 are not commensurate.

(iv) For $\mathrm{Ra} \geqslant \mathrm{Ra}_4$ the preceding biperiodic regime undergoes a transition of the type described above (intermittency).

Thus in both cases a continuous transition from a periodic or quasiperiodic regime to turbulent convection is observed.

Such a transition also occurs in the Belousov–Zhabotinski chemical reaction as observed by Roux *et al.*[6] The experiment is performed in a well-stirred tank; therefore no spatial structure can appear and the system's evolution is governed by the sole kinetic equations. That difference does not prevent the occurrence of a transition from an oscillatory regime to turbulence by intermittency when the flux of reacting products is increased. Therefore we may expect such a bifurcation to occur in model dynamical systems. Y. Pomeau and P. Manneville[7] have devised such dynamical systems and called type-I intermittency the transition to chaos which occurs for a one-parameter family of order-preserving endomorphisms of S^1 when a stable periodic orbit vanishes. That vanishing occurs when the stable orbit collapses with a nearby unstable one of the same period. Such a bifurcation is structurally stable: nearby families for the relevant C^r topology will undergo the same bifurcation. In what follows, we study a family $\{f_\varepsilon, \varepsilon \in]\varepsilon_c, \varepsilon_d[\}$of nonsingular order-preserving C^3 endomorphisms of S^1 with degree 2 displaying such a bifurcation. For $\varepsilon \leqslant 0$ the endomorphisms possess a stable fixed point x_ε^* (see Fig. 1). The bifurcation takes place at $\varepsilon = 0$ and leads to an Axiom A dynamical system (as defined by Z. Nitecki for endomorphisms of S^1 [8]).

Let x_0^* be the location of the stable fixed point for $\varepsilon = 0$. Then, the generic form of the endomorphism f_ε will be near x_0^* and for $|\varepsilon|$ small enough

$$f_\varepsilon(x) = x + \alpha\varepsilon + \frac{f_0^{(2)}(x_0^*)}{2}(x - x_0^*)^2 + \mathcal{O}(|x - x_0^*|(\varepsilon + (x - x_0^*)^2)) \qquad \alpha > 0$$

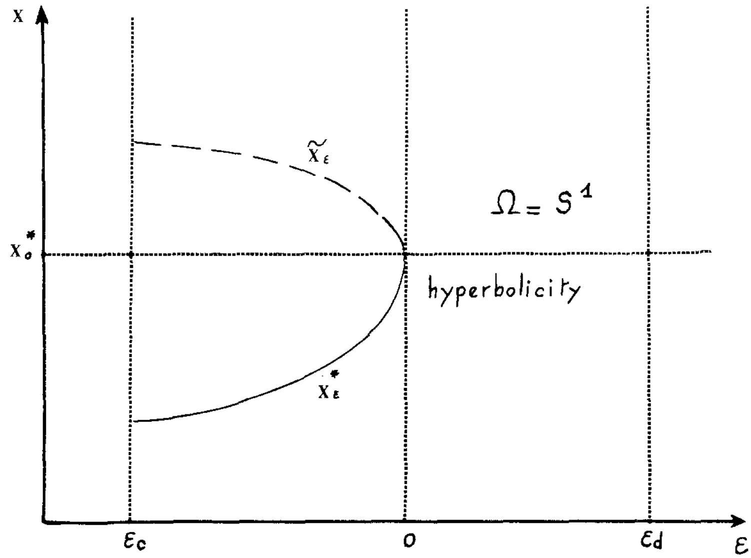

Fig. 1. Bifurcation diagram.

We take the precise form

$$f_\varepsilon(x) = x_0^* + \frac{x - x_0^* + \varepsilon}{1 - a(x - x_0^*)}$$

which enables explicit calculations (see Proposition 2.2 and Proposition 2.3). Far from x_0^*, we shall also require for the sake of simplicity that f_ε have a constant derivative.

In Section 2, we introduce our model and study its fundamental properties. We prove that for $\varepsilon > 0$, the dynamical system is hyperbolic and investigate the consequences of that hyperbolic character on the structure of fundamental intervals. We also deduce from the local form of the map f_ε near x_0^* some useful results about the orbits' behavior in that region.

In Section 3, we study the relevant invariant measures. We prove that for $\varepsilon \leqslant 0$, the relevant invariant measure is a Dirac measure. To do so, we prove by methods similar to Misiurewicz's[9] that almost every point belongs to the basin of the stable fixed point.

In Section 4, we study the continuity of the bifurcation from a measure-theoretical point of view. We show that for $\varepsilon > 0$ the absolutely continuous invariant measure is weakly continuous and prove that it converges to the Dirac measure $\delta^*_{x_0}$ for the weak topology when ε goes to zero. Those results imply that the expectation of any C^0 function continuously varies with ε, though a bifurcation take place. We also prove the continuity of the entropy

$h(\mu_\varepsilon)$ of the invariant measure. Those results give a striking example of the importance of measure-theoretical aspects in the study of dynamical systems. The bifurcation undergone by our system is continuous from a measure-theoretical point of view, though it is discontinuous from a topological one. Moreover measure-theoretical properties give us a much more detailed picture of our dynamical system than the crude topological description does.

The proof we give of the measure-theoretical continuity of the transition is mainly based on (1) the hyperbolicity of the system for $\varepsilon > 0$ and (2) the local expression of the map near x_0^*. We use the explicit form of f_ε only to prove the hyperbolicity and to obtain certain estimates we need (see Section 4.2). Therefore, we expect that every similar family of endomorphisms which displays a transition from a stable periodic orbit to an expanding regime will present the same property of continuity, of course more lengthy calculations would then be required to obtain estimates similar to those of Section 4.2.

2. THE MODEL AND ITS BASIC PROPERTIES

2.1. The Model

In this section, we define the model we use in the sequel. Let x_0^* be come real number in $]0, 1/2[$, a, ε_d and ε_c some real numbers $a > 0$, $\varepsilon_d < (3/4)x_0^*$ and $0 > \varepsilon_c > \sup\{-(1/a), -ax_0^{*2}, -(1/2)(a + 1 - 3ax_0^* + 2ax_0^{*2})\}$.

For any ε in $]\varepsilon_c, \varepsilon_d[$ we define

$$x_{1,\varepsilon} = \frac{1}{4a}\{1 + 3ax_0^* - [(1 - ax_0^*)^2 + 8a(x_0^* - \varepsilon)]^{1/2}\}$$

$$x_{2,\varepsilon} = \frac{1}{4a}\{1 + 3ax_0^* + [(1 - ax_0^*)^2 + 8a(x_0^* - \varepsilon)]^{1/2}\}$$

$$x_{3,\varepsilon} = x_0^* + \frac{1}{a}\left[1 + \left(\frac{1 + a\varepsilon}{2}\right)^{1/2}\right]$$

Then, we define a C^0 map f_ε from the interval $[0, 1]$ onto itself as follows (see Fig. 2):

$$\hat{f}_\varepsilon(x) = 2x \qquad \text{on } [0, x_{1,\varepsilon}]$$

$$\hat{f}_\varepsilon(x) = x_0^* + \frac{x - x_0^* + \varepsilon}{1 - a(x - x_0^*)} \qquad \text{on } [x_{1,\varepsilon}, x_{2,\varepsilon}]$$

$$\hat{f}_\varepsilon(x) = 2x \qquad \text{on } [x_{2,\varepsilon}, 1/2]$$

$$\hat{f}_\varepsilon(x) = 2x - 1 \qquad \text{on } [1/2, 1]$$

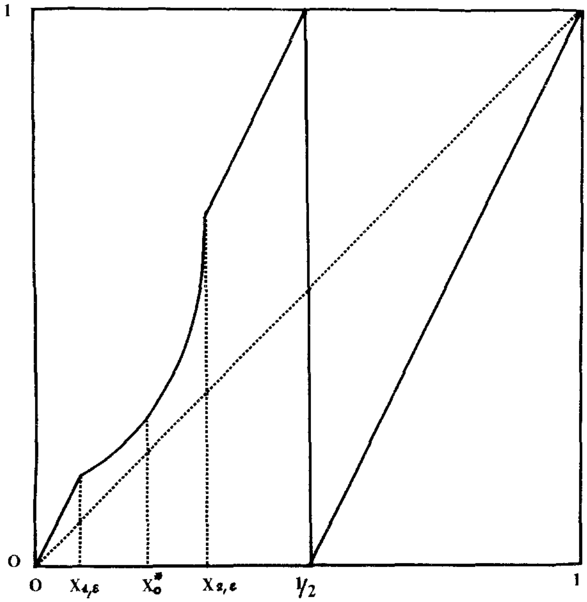

Fig. 2. The map $\hat{f}_\varepsilon$ (for $\varepsilon > 0$).

The definition of ε_c and ε_d ensures that

$$0 < x_{1,\varepsilon} < x_0^* < x_{3,\varepsilon} < x_{2,\varepsilon} < 1/2$$

Notice that $a = \frac{1}{2}(\hat{f}_0^{(2)}(x_0^*))$ and $\hat{f}_\varepsilon^{(1)}(x_{3,\varepsilon}) = 2$.

By a slight alteration of $\hat{f}_\varepsilon$ in neighborhoods of $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$, we transform $\hat{f}_\varepsilon$ into a C^3 map f_ε on S^1. We proceed to that regularization as follows. We choose some real numbers b and c $0 < b < x_{1\varepsilon} < c < x_0^*$ such that for every ε_c in $]\varepsilon_d, \varepsilon[$ $f_\varepsilon(b) > c$. Then we modify f_ε on $[b, c]$ so as to obtain a C^3 map which satisfies

(1) $f_\varepsilon^{(1)} > 1$ on $]b, x_{1,\varepsilon}[$

(2) $1 > f_\varepsilon^{(1)} > \hat{f}_\varepsilon^{(1)} > 0$ on $]x_{1,\varepsilon}, c[$ (see Fig. 3)

In the same way we choose some real numbers e and f, $x_{3,\varepsilon} < e < x_{2,\varepsilon} < f < 1/2$, such that for every ε in $]\varepsilon_c, \varepsilon_d[$ $\hat{f}_\varepsilon(e) > f$ and we modify $\hat{f}_\varepsilon$ on $[e, f]$ so as to obtain a C^3 map which satisfies $f_\varepsilon^{(1)} > 2$ on $]e, f[$ (see Fig. 4).

Moreover we proceed to the regularization so that the family $\{f_\varepsilon, \varepsilon \in\ ]\varepsilon_c, \varepsilon_d[\}$ satisfies the following properties:

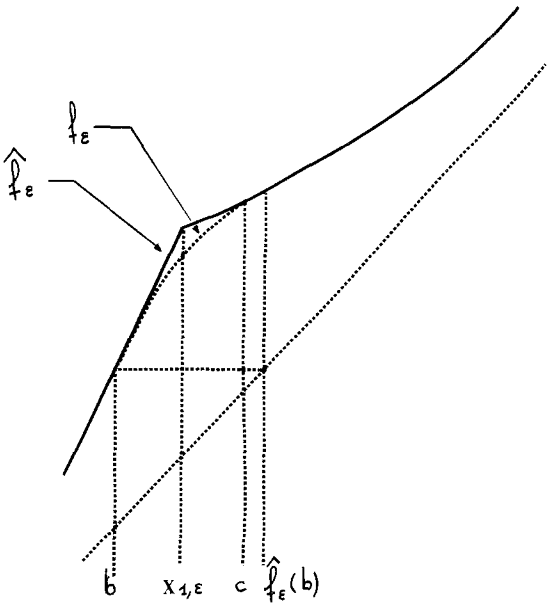

Fig. 3. Regularization near $x_{1,\varepsilon}$.

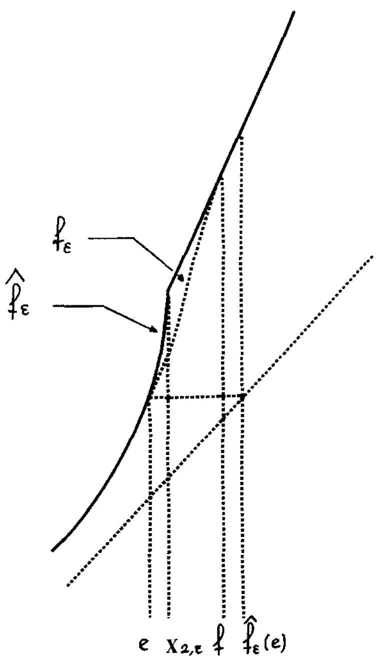

Fig. 4. Regularization near $x_{2,\varepsilon}$.

(1) If we call φ_ε the inverse map of f_ε from $[0, 1]$ onto $[0, 1/2]$ and ψ_ε the inverse map from $[0, 1]$ onto $[1/2, 1]$, for every point x in S^1, $f_\varepsilon(x)$, $\varphi_\varepsilon(x)$, and $f_\varepsilon^{(2)}(x)$ are C^1 functions of the parameter ε, while $f_\varepsilon^{(1)}$ is a C^2 function of ε.

(2) There is some real number $\omega > 0$ such that for every ε in $]\varepsilon_c, \varepsilon_d[f_\varepsilon^{(1)}(x) > \omega$. Therefore f_ε is a nonsingular map for every ε in $]\varepsilon_c, \varepsilon_d[$.

Since we can choose $c - b$ and $f - e$ as small as we wish (for $|\varepsilon_c|$ and ε_d small enough) and since the regularization has no other aim than ensuring the smoothness properties of the dynamical system, we shall no longer take account of it in what follows.

The dynamical system thus obtained has for $\varepsilon \leqslant 0$ a stable fixed point $x_\varepsilon^* \in]x_{1,\varepsilon}, x_{2,\varepsilon}[$. That fixed point vanishes for $\varepsilon > 0$. In that parameter range, the system is actually expanding. Before proving it, we shall prove some useful equalities. Those equalities are derived from the way we define f_ε in the region $[x_{1,\varepsilon}, x_{2,\varepsilon}]$.

2.2. Some Useful Equalities

Proposition 2.2.1. $\varphi_\varepsilon(y) = x_0^* + (y - x_0^* - \varepsilon)/[1 + a(y - x_0^*)]$ on the interval $[f_\varepsilon(x_{1,\varepsilon}), f_\varepsilon(x_{2,\varepsilon})]$.

Proposition 2.2.2. Let ε be in $]0, \varepsilon_d[$ and x be any point in $[x_{1,\varepsilon}, x_{2,\varepsilon}]$. Let $n \in \mathbb{N}$ be such that $f_\varepsilon^j(x) \leqslant x_{2,\varepsilon}$ for any j, $0 \leqslant j \leqslant n$. Then for any j, $0 \leqslant j \leqslant n$,

$$(1)\quad f_\varepsilon^j(x) = x_0^* + \left(\frac{\varepsilon}{a}\right)^{1/2} \tan\left\{ j \arctan(a\varepsilon)^{1/2} + \arctan\left[\left(\frac{a}{\varepsilon}\right)^{1/2} (x - x_0^*)\right]\right\}$$

$$(2)\quad (f_\varepsilon^j)^{(1)}(x) = \frac{\varepsilon + a(f_\varepsilon^j(x) - x_0^*)^2}{\varepsilon + a(x - x_0^*)^2}$$

The proof is recursive. It is left to the reader.

2.3. Hyperbolicity of the Dynamical System for $\varepsilon > 0$

Theorem 2.3. If $ax_0^* < 2/3$ and ε_d is small enough, then the dynamical system is expanding in the following sense: there is some $d > 1$ and a continuous positive function $F(\varepsilon)$ on $]0, \varepsilon_d[$ which does not vanish on that interval such that for any ε in $]0, \varepsilon_d[$ and any $n \in \mathbb{N}$

$$\operatorname*{Inf}_{S^1} (f_\varepsilon^n)^{(1)}(x) > F(\varepsilon) d^{n\sqrt{\varepsilon}}$$

The proof is given in Appendix 2.3.

2.4. The Fundamental Intervals

For any ε in $]0, \varepsilon_d[$ we call fundamental interval of order k any connected closed subset $Ik, \bar{i}, \varepsilon$ which satisfies $f_\varepsilon^k(Ik, \bar{i}, \varepsilon) = [0, 1]$ $\bar{i} = (i_0, ..., i_{k-1}) \in \{0, 1\}^k$ is the kneading sequence of $Ik, \bar{i}, \varepsilon$. It is thus defined: $i_n = 0$ if and only if $f_\varepsilon^n(Ik, \bar{i}, \varepsilon) \subset [0, 1/2]$, $i_n = 1$ otherwise.

In this part, we derive from the hyperbolicity two results on fundamental intervals we shall use later:

(i) First we give an upper bound for the Lebesgue measure of any fundamental interval (Proposition 2.4.1). [We shall denote by $\lambda(A)$ the Lebesgue measure of the measurable set A.]

(ii) Then we prove that the Lebesgue measure of any fundamental interval is a continuous function of ε (Lemma 2.4.2).

Proposition 2.4.1. For any ε in $]0, \varepsilon_d[$, any $k \in \mathbb{N}$, any fundamental interval $Ik, \bar{i}, \varepsilon$, $\lambda(Ik, \bar{i}, \varepsilon) < [1/F(\varepsilon)]\, d^{-k\sqrt{\varepsilon}}$.

Proof. Since $f_\varepsilon^k(Ik, \bar{i}, \varepsilon) = [0, 1]$,

$$\lambda(Ik, \bar{i}, \varepsilon) < \frac{1}{\operatorname{Inf}_{S^1}(f_\varepsilon^k)^{(1)}}$$

Using Theorem 2.3 we have then

$$\lambda(Ik, \bar{i}, \varepsilon) < \frac{1}{F(\varepsilon)}\, d^{-k\sqrt{\varepsilon}} \qquad \blacksquare$$

Lemma 2.4.2. There is some continuous function χ on $]0, \varepsilon_d[$ such that for any given ε in $]0, \varepsilon_d[$, $|\lambda(Ik, \bar{i}, \varepsilon') - \lambda(Ik, \bar{i}, \varepsilon)| < \chi(\varepsilon)\, |\varepsilon' - \varepsilon|$ holds, for every $k \in \mathbb{N}$, every kneading sequence $\bar{i} \in \{0, 1\}^k$ and any ε' in some neighborhood of ε.

The proof is given in Appendix 2.4.2.

2.5. The Nature of the Map near x_0^* and Its Consequences

We derive from the nature of the map near x_0^* some results about the orbits of points in that region.

Lemma 2.5.1. We may choose some $\beta > 0$ such that for ε_d small enough, we have the following:

(1) For any x_0 in $]x_0^* + \beta/4,\ x_0^* + \beta/2[$, any ε in $]0, \varepsilon_d[$, and any

$n \in \mathbb{N}$ such that $n \leqslant \beta\varepsilon^{-1/2}$ the sequence $\{x_p, p \in \{0,..., n\}\}$ defined by $x_{p+1} = \varphi_\varepsilon(x_p)$ satisfies

$$\left| x_p - x_0^* - \frac{1}{a(p+\tau)} \right| \leqslant \frac{1}{(p+\tau)^{3/2}}$$

where $\tau = 1/a(x_0 - x_0^*)$.

(2) For any x_0 in $]x_0^* - \beta/2, x_0^* - \beta/4[$, any ε in $]0, \varepsilon_d[$, and any $n \in \mathbb{N}$ such that $n \leqslant \beta\varepsilon^{-1/3}$, the sequence $\{x_p, p \in \{0,..., n\}\}$ defined by $x_{p+1} = f_\varepsilon(x_p)$ satisfies

$$\left| x_0^* - x_p - \frac{1}{a(p+\tau')} \right| \leqslant \frac{1}{(p+\tau')^{3/2}} \qquad \text{where} \quad \tau' = \frac{1}{a(x_0^* - x_0)}$$

The proof is given in Appendix 2.5.1.

We extend this result to $\varepsilon = 0$.

Lemma 2.5.2. For β small enough and any x_0 in $]x_0^*, x_0^* + \beta[$ the sequence $\{x_n, n \in \mathbb{N}\}$ defined by $x_{n+1} = \varphi_0(x_n)$ satisfies

$$\left| x_n - x_0^* - \frac{1}{a(n+\tau)} \right| \leqslant \frac{1}{(n+\tau)^{3/2}} \qquad \text{where} \quad \tau = \frac{1}{a(x_0 - x_0^*)}$$

for every $n \in \mathbb{N}$.

The proof is the same as before, except that the Proposition 2.5.4 of Appendix 2.5.1 is now pointless.

We shall also need in Appendix 3.2.1 the following result.

Lemma A.5.3. For β small enough and any η in $[x_0^*, x_0^* + \beta]$

(1) $f_0^{(2)}(x) > 0$ on $[x_0^*, \eta]$

(2) $\sup_{[x_0^*, \eta]} f_0^{(1)}(x) \leqslant \inf_{[\eta, 1/2]} f_0^{(1)}(y)$

The proof is left to the reader.

3. INVARIANT MEASURES

We require for an invariant measure μ_ε to be physically meaningful that for every C^0 function g on S^1 and almost every point x

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} g(f_\varepsilon^j(x)) = \mu_\varepsilon(g)$$

Such measures are usually called Bowen–Ruelle measures.[10]

In what follows we prove that for ε in $]\varepsilon_c, 0[$, the Dirac measure $\delta(x_\varepsilon^*)$ is our system's Bowen–Ruelle measure (Section 3.1). We extend that result to $\varepsilon = 0$, which requires some modifications of the preceding proof (Section 3.2). Then in Section 3.3, we state some well-known results about the Bowen–Ruelle measure of a hyperbolic system (such as ours for ε in $]0, \varepsilon_d[$); we shall use those results in Section 3 to prove the weak continuity of the invariant measure and the continuity of its metric entropy.

3.1. $\epsilon < 0$

When $\varepsilon_c < \varepsilon < 0$, the map f_ε has a linearly stable fixed point x_ε^*. S^1 is divided into two subsets: (1) the basin of x_ϵ^* which is an open set, and (2) an invariant cantor set C_ε. The Dirac measure concentrated at x_ε^* is an invariant measure. We show in this part that it is the Bowen–Ruelle measure by proving that $\lambda(C_\varepsilon) = 0$ (Theorem 3.1)

We first introduce some notations:

(i) $\tilde{x}_\varepsilon$ is the unstable fixed point lying in $]x_\varepsilon^*, 1/2[$.

(ii) $I_\varepsilon =]0, \tilde{x}_\varepsilon[$.

(iii) $J_\varepsilon =]1/2, \psi_\varepsilon(\tilde{x}_\varepsilon)[$.

For every $n \in \mathbb{N}$, we define $E_n = \{x \in C_{S^1} I_\varepsilon,\ f_\varepsilon^p(x) \notin J_\varepsilon$ for every p $0 \leqslant p \leqslant n\}$ (see Fig. 5).

E_n is the set of points, the $n+1$ first iterates of which remain outside I_ε. $\{E_n, n \in \mathbb{N}\}$ is obviously a decreasing sequence of subsets and the Cantor set C_ε may be defined as $C_\varepsilon = \bigcap_{n \in \mathbb{N}} E_n$.

We are going to prove (Lemma 3.1.4) that $\{\lambda(R_n), n \in \mathbb{N}\}$ is majorized by a geometric sequence which converges to zero. The result $\lambda(C_\varepsilon) = 0$ is a direct consequence of that lemma.

Before proving Lemma 3.1.4 we need some preliminary results. Let K_n be any connected component of E_n. Then, we have the following:

Lemma 3.1.1. The map $f_\varepsilon^n|_{K_n}$ is a diffeomorphism from K_n onto $f_\varepsilon^n(K_n)$.

The proof is inductive.

We have also the following:

Proposition 3.1.2. $f_\varepsilon^n(K_n \backslash E_{n+1})$ is equal to either $\varphi_\varepsilon(J_\varepsilon)$ or $\psi_\varepsilon(J_\varepsilon)$.

Proof. The recursive proof is left to the reader.

Lemma 3.1.3. There is some $\delta > 1$ such that for every $n \in \mathbb{N}$, every

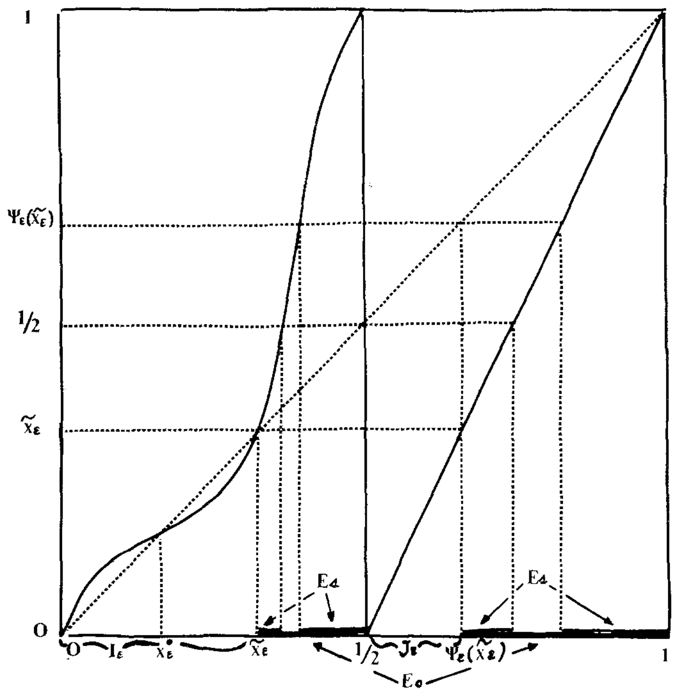

Fig. 5. Subsets E_n for $\varepsilon < 0$.

connected component K_n of E_n and any point (a, b) in K_n^2 the following inequality holds:

$$\frac{1}{\delta} \leqslant \frac{(f_\varepsilon^n)^{(1)}(a)}{(f_\varepsilon^n)^{(1)}(b)} \leqslant \delta$$

Proof. We set $\alpha = \inf_{C_{S^1} I_\varepsilon} f_\varepsilon^{(1)}(x)$, $\alpha > 1$, and $L = \sup_{C_{S^1} I_\varepsilon} |f_\varepsilon^{(2)}(x)|$. Then

$$\sup_{C_{S^1} I_\varepsilon} |(\log f_\varepsilon^{(1)})^{(1)}| \leqslant \frac{\sup_{C_{S^1} I_\varepsilon} |f_\varepsilon^{(2)}|}{\inf_{C_{S^1} I_\varepsilon} f_\varepsilon^{(1)}} \leqslant L$$

For a given n and given (a, b) in K_n^2,

$$\left| \log \left\{ \frac{(f_\varepsilon^n)^{(1)}(a)}{(f_\varepsilon^n)^{(1)}(b)} \right\} \right| \leqslant \sum_{k=0}^{n-1} |\log f_\varepsilon^{(1)}(f_\varepsilon^k(a)) - \log f_\varepsilon^{(1)}(f_\varepsilon^k(b))|$$

Hence

$$\left| \log \left\{ \frac{(f_\varepsilon^n)^{(1)}(a)}{(f_\varepsilon^n)^{(1)}(b)} \right\} \right| \leqslant L \sum_{k=0}^{n-1} |f_\varepsilon^k(a) - f_\varepsilon^k(b)|$$

From Proposition 3.1.1 we have $|f_\varepsilon^n(a) - f^n(b)| < 1$. Hence for every k $0 \leqslant k \leqslant n-1$, $|f_\varepsilon^{n-k}(f_\varepsilon^k(a)) - f_\varepsilon^{n-k}(f_\varepsilon^k(b))| < 1$, which implies $|f_\varepsilon^k(a) - f_\varepsilon^k(b)| \cdot \alpha^{n-k} \leqslant 1$. Therefore

$$\left| \log \left\{ \frac{(f_\varepsilon^n)^{(1)}(a)}{(f_\varepsilon^n)^{(1)}(b)} \right\} \right| \leqslant L \sum_{k=0}^{n-1} \frac{1}{\alpha^{n-k}}$$

and, setting $\delta = \exp[L/(\alpha - 1)]$, we have

$$\frac{1}{\delta} \leqslant \frac{(f_\varepsilon^n)^{(1)}(a)}{(f_\varepsilon^n)^{(1)}(b)} \leqslant \delta \qquad \blacksquare$$

Lemma 3.1.4. There is some $\zeta > 0$ such that for every $n \in \mathbb{N}$,

$$\lambda(E_{n+1}) \leqslant (1 - \zeta)\, \lambda(E_n)$$

Proof. We have

$$\lambda(K_n) \inf_{K_n} (f_\varepsilon^n)^{(1)} \leqslant \lambda(f_\varepsilon^n(K_n)) \leqslant 1 \tag{3.1.5}$$

and also

$$\lambda(f_\varepsilon^n(K_n \backslash E_{n+1}) \leqslant \lambda(K_n \backslash E_{n+1}) \sup_{K_n} (f_\varepsilon^n)^{(1)} \tag{3.1.6}$$

if we set $\xi = \inf\{\lambda(\varphi_\varepsilon(J_\varepsilon)), \lambda(\psi_\varepsilon(J_\varepsilon))\}$, we have from Proposition 3.1.2

$$\lambda(f_\varepsilon^n(K_n \backslash E_{n+1})) \geqslant \xi \tag{3.1.7}$$

From inequalities 3.1.6 and 3.1.7, we deduce

$$\lambda(K_n \backslash E_{n+1}) \geqslant \frac{\xi}{\sup_{K_n}(f_\varepsilon^n)^{(1)}} \tag{3.1.8}$$

From inequalities 3.1.5 and 3.1.8 we have then

$$\frac{\lambda(K_n \cap E_{n+1})}{\lambda(K_n)} = 1 - \frac{\lambda(K_n \backslash E_{n+1})}{\lambda(K_n)} \leqslant 1 - \xi \frac{\inf_{K_n}(f_\varepsilon^n)^{(1)}}{\sup_{K_n}(f_\varepsilon^n)^{(1)}}$$

From Lemma 3.1.3, we have then

$$\frac{\lambda(K_n \cap E_{n+1})}{\lambda(K_n)} \leqslant 1 - \frac{\xi}{\delta}$$

Thus setting $\zeta = \xi/\delta$, we obtain

$$\lambda(K_n \cap E_{n+1}) \leqslant (1-\zeta)\,\lambda(K_n)$$

and summing over connected components of E_n

$$\lambda(E_{n+1}) \leqslant (1-\zeta)\,\lambda(E_n) \qquad \blacksquare$$

Theorem 3.1. $\lambda(C_\varepsilon) = 0$ for every ε in $]\varepsilon_c, 0[$.

Proof. For every $n \in \mathbb{N}$, $C_\varepsilon \subset E_n$ and $\lambda(C_\varepsilon) \leqslant \lambda(E_n) \leqslant (1-\zeta)^n\,\lambda(E_0)$. Therefore $\lambda(C_\varepsilon) = 0$. $\blacksquare$

3.2. $\epsilon = 0$

In this section, we extend the preceding results to $\varepsilon = 0$ by proving that $\lambda(C_0) = 0$. The map f_0 is not expanding on $[x_0^*, 1]$ since $f_0^{(1)}(x_0^*) = 1$. Therefore we must modify the preceding proof, which relied on the fact that $f_\varepsilon^{(1)}(x) > \alpha > 1$ on $[\tilde{x}_\varepsilon, 1]$ and we must now take into account the nature of the map near x_0^*.

As in Section 3.1, we consider the subsets E_n. For every n, x_0^* is the left end point of a particular connected component I_n of E_n. It plays a special part since the map f_0 is expanding on every component of E_n except I_n. Besides $I_{n+1} = \varphi_0(I_n)$ (see Fig. 6). We now prove a result similar to Lemma 3.1.3:

Lemma 3.2.1. There is some $\delta > 1$ such that for any $n \in \mathbb{N}$, for any connected component K_n of E_n except I_n, and for any point (a, b) in K_n^2,

$$\frac{1}{\delta} \leqslant \frac{(f_0^n)^{(1)}(a)}{(f_0^n)^{(1)}(b)} \leqslant \delta$$

Proof. We give the proof in Appendix 3.2.1.

Then we consider the components I_n and prove that their Lebesgue measure goes to zero when n goes to infinity.

Lemma 3.2.2. There is some $N_0 \in \mathbb{N}$ such that for $n \geqslant N_0$

$$\lambda(I_n) \leqslant \frac{2}{a(n - N_0 + 1)}$$

Proof. We know that $I_n = \varphi_0^n(I_0)$. Therefore $\lambda(I_n)$ goes to zero when n goes to infinity and for any given β there is some N_0 such that

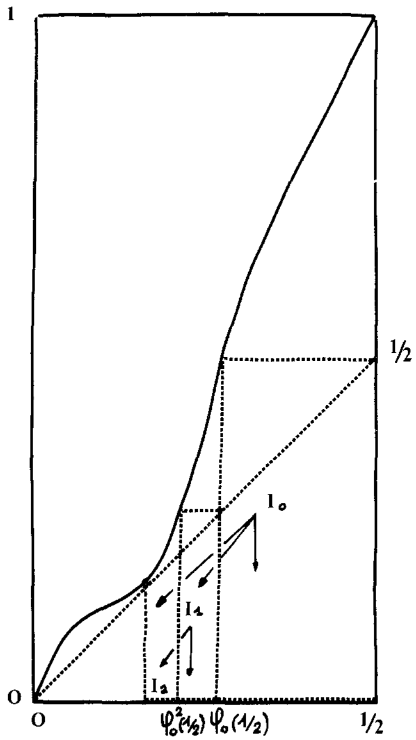

Fig. 6. Connected components I_n for $\varepsilon = 0$.

$I_{N_0} \subset [x_0^*, x_0^* + \beta]$. Then we apply Lemma 2.5.5 to the sequence $\{x_n, n \in \mathbb{N}\}$ defined by $x_0 = \sup\{x, x \in I_{N_0}\}$ and we get

$$\lambda(I_{N_0+n}) \leqslant \frac{1}{a(n+\tau)} + \frac{1}{(n+\tau)^{3/2}}, \qquad \text{where} \quad \tau = \frac{1}{a\lambda(I_{N_0})}$$

Therefore, we have for β small enough

$$\lambda(I_{N_0+n}) \leqslant \frac{2}{a(n+1)} \qquad \blacksquare$$

Now we prove the result we were aiming at and which is a consequence of the two preceding results:

Theorem 3.2. $\lambda(C_0) = 0$.

Proof. Let K_{N_0+n} be any component of E_{N_0+n} such that $K_{N_0+n} \cap I_{N_0+n} = \varnothing$. From Lemma 3.2.1, we deduce, as in the proof of Lemma 3.1.4,

$$\lambda(K_{N_0+n} \cap E_{N_0+n+1}) \leqslant (1 - G)\,\lambda(K_{N_0+n})$$

where $G = (1/\delta)\inf\{\lambda(\varphi_0 \circ \psi_0([0, x_0^*])), \lambda(\psi_0^2([0, x_0^*]))\} < 1$. Since

$$\lambda(E_{N_0+n+1}) = \lambda(I_{N_0+n} \cap E_{N_0+n+1}) + \sum_{\substack{K_{N_0+n} \\ K_{N_0+n} \cap I_{N_0+n} = \varnothing}} \lambda(K_{N_0+n} \cap E_{N_0+n+1})$$

we have

$$\lambda(E_{N_0+n+1}) \leqslant \lambda(I_{N_0+n}) + (1-G)\,\lambda(E_{N_0+n})$$

Using Lemma 3.2.2 we obtain

$$\lambda(E_{N_0+n+1}) \leqslant \frac{2}{a(n+1)} + (1-G)\,\lambda(E_{N_0+n})$$

so that

$$\lambda(E_{N_0+n}) \leqslant (1-G)^n\,\lambda(E_{N_0}) + \frac{2}{a}\sum_{p=0}^{n-1}\frac{(1-G)^p}{n-p}$$

We have for $n > 1$

$$\begin{aligned}\sum_{p=0}^{n-1}\frac{(1-G)^p}{n-p} &= \sum_{p=0}^{[n/2]}\frac{(1-G)^p}{n-p} + \sum_{p=[n/2]+1}^{n-1} \leqslant \frac{2}{n}\sum_{p=0}^{\infty}(1-G)^p \\ &\quad + (1+G)^{n/2}\sum_{p=[n/2]+1}^{n-1}\frac{1}{n-p} \leqslant \frac{2}{nG} + n(1-G)^{n/2}\end{aligned}$$

(Throughout this text $[\alpha]$ denotes the greatest integer which is lesser or equal to α.) Therefore, we have

$$\lambda(E_{N_0+n}) \leqslant (1-G)^n\,\lambda(E_{N_0}) + \frac{4}{aGn} + \frac{2n}{a}(1-G)^{n/2}$$

When n goes to infinity, the right-hand side goes to zero. And, since for every $n \in \mathbb{N}$ $C_0 \subset E_{N_0+n}$, we conclude that $\lambda(C_0) = 0$. ∎

Therefore the Bowen–Ruelle measure is for $\varepsilon = 0$, the Dirac measure $\delta(x_0^*)$

3.3. $\epsilon > 0$

For ε in $]0, \varepsilon_d[$ the map f_ε is expanding. Such systems have been widely studied. So we merely state the results which are useful to us:

(1) The system has one and only one absolutely continuous invariant measure μ_ε.

(2) The density ρ_ε of that invariant measure is a C^1 function on S^1 since f_ε is C^3.

(3) On $\sup(\mu_\varepsilon) = \overline{\{x \in S^1,\ \rho_\varepsilon(x) \neq 0\}}$, ρ_ε is bounded far away from zero.

(4) The metric enropy $h(\mu_\varepsilon)$ of the invariant measure satisfies

$$h(\mu_\varepsilon) = \mu_\varepsilon(\log f_\varepsilon^{(1)})$$

4. CONTINUITY OF THE INVARIANT MEASURE

We have shown that for $\varepsilon \leqslant 0$, the Ruelle–Bowen invariant measure is a Dirac measure. Its location, the stable fixed point x_ε^*, smoothly depends on ε. Therefore, the Bowen–Ruelle measure is weakly continuous on $]\varepsilon_c, 0[$. In this part, we prove that the invariant measure is continuous for the weak topology when $\varepsilon > 0$ (Section 4.1). Then we prove the weak continuity at the bifurcation threshold $\varepsilon = 0$ and conclude that the Bowen–Ruelle measure is a continuous function of ε on $]\varepsilon_c, \varepsilon_d[$ for the weak topology (Section 4.2). We also prove the continuity of the Kolmogoroff–Sinaï entropy (Section 4.3).

4.1. Weak Continuity of μ_ϵ for $\epsilon > 0$

In this section we prove

Theorem 4.1. For any C^0 function g on $[0, 1]$, the expectation $\mu_\varepsilon(g)$ is a continuous function of ε on $]0, \varepsilon_d[$.

We use symbolic dynamics related to the Markov partition $\{[0, 1/2], [1/2, 1]\}$ to prove that result. $(\{0, 1\}^{\mathbb{N}}, d)$ is a compact space for the metric d thus defined: For $(\bar{i}, \bar{j}) \in \{0, 1\}^{\mathbb{N}} \times \{0, 1\}^N$, let $k = \sup\{n \in \mathbb{N}, \forall p \in \mathbb{N}, p < n, i_p = j_p\}$, then $d(\bar{i}, \bar{j}) = 1/2^k$.

For every ε in $]0, \varepsilon_d[$ we define a map π_ε from $\{0, 1\}^{\mathbb{N}}$ onto $[0, 1]$ by setting $\pi_\varepsilon(\bar{i}) = x$, if $\bar{i}$ is an itinerary of x, that is to say, if $i_n = [f_\varepsilon^n(x) + 1/2]$ for every $n \in \mathbb{N}$. The map π_ε is not one-to-one since pre-images of 1/2 have two itineraries. However the Lebesgue measure of this set is zero so we shall encounter no trouble. We prove the following:

Lemma 4.1.1. π_ε is Hölder continuous:

$$|\pi_\varepsilon(\bar{i}) - \pi_\varepsilon(\bar{j})| < \frac{d\sqrt{\varepsilon}}{F(\varepsilon)}\,(d(\bar{i}, \bar{j}))^{(\log d/\log 2)\sqrt{\varepsilon}}$$

Proof. Assume $d(\bar{i}, \bar{j}) = 1/2^k$ for some $k \geqslant 1$. Then $\pi_\varepsilon(\bar{i})$ and $\pi_\varepsilon(\bar{j})$ lie in the same fundamental interval of order $k-1$, so that from 2.4.1 we have

$$|\pi_\varepsilon(\bar{i}) - \pi_\varepsilon(\bar{j})| < \frac{1}{F(\varepsilon)} d^{-(k-1)\sqrt{\varepsilon}} \leqslant \frac{d\sqrt{\varepsilon}}{F(\varepsilon)} (d(\bar{i}, \bar{j}))^{(\log d)/\log 2)\sqrt{\varepsilon}}$$

and this inequality still holds if $d(\bar{i}, \bar{j}) = 1$. ∎

Moreover the map $\pi: \varepsilon \to \pi_\varepsilon$ from $]0, \varepsilon_d[$ into $\mathscr{C}^0(\{0, 1\}^{\mathbb{N}}, [0, 1])$ satisfies a local Lipschitz property:

Lemma 4.1.2. There is some continuous function χ on $]0, \varepsilon_d[$ such that for every ε_0 in $]0, \varepsilon_d[$, $|\pi_\varepsilon(\bar{i}) - \pi_{\varepsilon_0}(\bar{i})| < \chi(\varepsilon_0) |\varepsilon - \varepsilon_0|$ on a neighborhood $V(\varepsilon_0)$ of ε_0.

Proof. That result is nothing else than Lemma 2.4.2 under an other guise.

To every C^0 function g on $[0, 1]$ we associate a C^0 function $\bar{g}_\varepsilon$ on $\{0, 1\}^{\mathbb{N}}$ thus defined $\bar{g}_\varepsilon(\bar{i}) = g \circ \pi_\varepsilon(\bar{i})$

Lemma 4.1.3. $\bar{g}_\varepsilon$ is continuous in ε on $]0, \varepsilon_d[$ for the C^0 norm.

This is an obvious consequence of Lemma 4.1.2. We define a measure $\bar{\lambda}_\varepsilon$ on $\{0, 1\}^{\mathbb{N}}$ by setting $\bar{\lambda}_\varepsilon(\bar{g}_\varepsilon) = \lambda(g)$ for every g in $\mathscr{C}^0([0, 1])$. We define a potential $\{\phi_\varepsilon^k, k \in \mathbb{N}\}$ on $\{0, 1\}^{\mathbb{N}}$ as follows:

Let $\mathscr{C}k$ be the set $\{(j,..., j+k), j \in \mathbb{N}\}$. For every $k \in \mathbb{N}$, we define a function Φ_ε^k on $\mathscr{C}k \times \{0, 1\}^{k+1}$ as follows:

$$\Phi_\varepsilon^0(j; i_0) = \log f_\varepsilon^{(1)} \circ \pi_\varepsilon(\sigma_1(i_0))$$

and for $k > 1$

$$\Phi_\varepsilon^k(j,..., j+k; i_0,..., i_k) = \log f_\varepsilon^{(1)} \circ \pi_\varepsilon \circ \sigma_{k+1}(i_0,..., i_k) - \log f_\varepsilon^{(1)} \circ \pi_\varepsilon \circ \sigma_k(i_0,..., i_{k-1})$$

where σ_j denotes the embedding from $\{0, 1\}^j$ into $\{0, 1\}^{\mathbb{N}}$ which associates to $(i_0,..., i_j)$ the sequence $\bar{l}$ of $\{0, 1\}^{\mathbb{N}}$ thus defined: $l_p = i_p$ for $0 \leqslant p \leqslant j$, $l_p = 0$ for $p > j$. Then $\bar{\lambda}_\varepsilon$ is the Gibbs measure on $\{0, 1\}^{\mathbb{N}}$ associated to the potential $\{\Phi_\varepsilon^k, k \in \mathbb{N}\}$ as was shown by O. Lanford.[11] $(\{0, 1\}^{\mathbb{Z}}, d)$ is a compact space for the metric d thus defined: for $(\tilde{i}, \tilde{j}) \in \{0, 1\}^{\mathbb{Z}} \times \{0, 1\}^{\mathbb{Z}}$, let $k = \sup\{n \in \mathbb{N}, \forall p \in \mathbb{Z}, |p| < n, i_p = j_p\}$, then $d(\tilde{i}, \tilde{j}) = 1/2^k$.

We define a shift-invariant potential $\{\psi_\varepsilon^k, k \in \mathbb{N}\}$ on $\{0, 1\}^{\mathbb{Z}}$ as follows:

For every $k \in \mathbb{N}$, we introduce $Dk = \{(j,..., j+k), j \in \mathbb{Z}\}$ and we define a function ψ_ε^k on $Dk \times \{0, 1\}^{k+1}$ as follows:

$$\psi_\varepsilon^k(j,..., j+k; i_0,..., i_k) = \Phi_\varepsilon^k(0,..., k; i_0,..., i_k)$$

The potential $\{\psi_\varepsilon^k,\ k \in \mathbb{N}\}$ is invariant under the shift automorphism on $\{0, 1\}^{\mathbb{Z}}$. Besides, there is some $D > 0$ such that for every $k \in \mathbb{N}$ and every ε in $]0, \varepsilon_d[$

$$\|\psi_\varepsilon^k\| = \sup_{(i_0,\dots,i_k)\in\{0,1\}^{k+1}} \psi_\varepsilon^k(0,\dots,k;\, i_0,\dots,i_k) < D$$

Moreover the potential decreases fast with increasing k:

Lemma 4.1.4. For any ε_0 in $]0, \varepsilon_d[$ there is some $\alpha > 0$ and some $B > 0$ such that the following inequality holds on a neighborhood $V(\varepsilon_0)$ of ε_0,

$$\sum_{k=0}^{\infty} k\, \|\psi_\varepsilon^k\|\, e^{k\alpha} < B$$

Proof. The function $\log f_{\varepsilon_0}^{(1)}$ is Lipschitz:

$$|\log f_{\varepsilon_0}^{(1)}(x) - \log f_{\varepsilon_0}^{(1)}(y)| < A_{\varepsilon_0}\, |x - y| \tag{4.1.4}$$

where

$$A_{\varepsilon_0} = \frac{\sup_{S^1} |f_{\varepsilon_0}^{(2)}|}{\inf_{S^1} |f_{\varepsilon_0}^{(1)}|}$$

We have $d(\sigma_k(i_0,\dots,k), \sigma_{k-1}(i_0,\dots,k-1)) \leqslant 1/2^k$. Therefore, from Lemma 4.1.1 we deduce

$$|\pi_{\varepsilon_0}(\sigma_k(i_0,\dots,k)) - \pi_{\varepsilon_0}(\sigma_{k-1}(i_0,\dots,k-1))| < \frac{d^{\varepsilon_0^{1/2}}}{F(\varepsilon_0)}\, 2^{-k\varepsilon_0^{1/2}(\log d/\log 2)}$$

Taking $x = \pi_{\varepsilon_0}(\sigma(i_0,\dots,i_k))$ and $y = \pi_{\varepsilon_0}(\sigma_{k-1}(i_0,\dots,k-1))$ in inequality 4.1.4 we obtain

$$\begin{aligned}&|\log f_{\varepsilon_0}^{(1)} \circ \pi_{\varepsilon_0}(\sigma_k(i_0,\dots,i_k)) - \log f_{\varepsilon_0}^{(1)} \circ \pi_{\varepsilon_0}(\sigma_{k-1}(i_0,\dots,i_{k-1}))| \\ &\quad < \frac{A_{\varepsilon_0} d^{(\varepsilon_0)^{1/2}}}{F(\varepsilon_0)}\, 2^{-k\varepsilon_0^{1/2}(\log d/\log 2)}\end{aligned}$$

so that

$$\|\psi_{\varepsilon_0}^k\| < \frac{A_{\varepsilon_0} d^{\varepsilon_0^{1/2}}}{F(\varepsilon_0)}\, 2^{-k\varepsilon_0^{1/2}(\log d/\log 2)}$$

If we set $\alpha = 2^{(\varepsilon_0^{1/2}/2)(\log d/\log 2)}$ and $B = 2A_{\varepsilon_0} d^{\varepsilon_0^{1/2}} \sum_0^\infty k(2^{-(\varepsilon_0^{1/2}/2)(\log d/\log 2)})^k$ we have then on a neighborhood $V(\varepsilon_0)$ of ε_0, $\sum_{k=0}^{\infty} k\, \|\psi_\varepsilon^k\|\, e^{k\alpha} < B$. ∎

We call $\tilde{\lambda}_\varepsilon$ the shift-invariant Gibbs masure on $\{0, 1\}^{\mathbb{Z}}$ associated with the potential $\{\psi^k_\varepsilon, k \in \mathbb{N}\}$. We define a shift-invariant measure $\bar{\mu}_\varepsilon$ on $\{0, 1\}^{\mathbb{N}}$ by $\bar{\mu}_\varepsilon(\bar{g}) = \tilde{\lambda}_\varepsilon(\bar{g} \circ \tau)$ for every C^0 function $\bar{g}$ on $\{0, 1\}^{\mathbb{N}}$.

Then we have the following:

Lemma 4.1.5. $\bar{\mu}_\varepsilon(\bar{g}_\varepsilon)$ is a C^0 function of ε on $]0, \varepsilon_d[$.

We give the proof in Appendix 4.1.5.

Then, since the absolutely continuous f_ε-invariant measure μ_ε on $[0, 1]$ satisfies $\mu_\varepsilon(g) = \bar{\mu}_\varepsilon(g_\varepsilon)$ for every C^0 function g on $[0, 1]$, Theorem 4.1 follows immediately.

4.2. Continuity of the Invariant Measure at the Bifurcation Threshold $\epsilon = 0$

4.2.1. Preliminary Study. In this section, we show that far from x_0^* the invariant density ρ_ε is well behaved (Theorem 4.2.1).

Theorem 4.2.1. For any $\gamma > 0$ small enough, there exists some $K' \geqslant 0$ such that for ε_d small enough, the density ρ_ε of the invariant measure satisfies

$$\rho_\varepsilon(x) \leqslant e^{K'|x-y|} \rho_\varepsilon(y)$$

for every ε in $]0, \varepsilon_d[$ and every x and y in $C_{S^1}]x_0^* - \gamma,\ x_0^* + \gamma[$. An immediate consequence of the result is that ρ_ε does not vanish.

The proof of Theorem 4.2.1 is as follows. We first prove two technical results (Lemma 4.2.1 and Lemma 4.2.2). From those results, we deduce that the nth iterate of the Lebesgue measure $h_{n,\varepsilon}$ is well behaved far from x_0^*. Then Lemma 4.2.1 enables us to take the limit thus proving Theorem 4.2.1.

Lemma 4.2.1. For ε_d and γ small enough, there exists some real number $C > 1$ such that for every ε in $]0, \varepsilon_d[$ and any interval I satisfying the following conditions:

(1) $\lambda(I) \leqslant \eta_0$ where $\eta_0 = a\gamma^2$.

(2) $I \cap J \neq \varnothing$ ($J =]x_0^* - \gamma, x_0^* + \gamma[$).

(3) For some $p > 1$, $f^p_\varepsilon(I) \cap J = \varnothing$ and for every q, $1 \leqslant q < p$ $f^q_\varepsilon(I) \cap J \neq \varnothing$.

(4) $\lambda(f^p_\varepsilon(I)) \leqslant \eta_0$.

We have

$$\sum_{t=1}^{p-1} \lambda(f^t_\varepsilon(I)) \leqslant C\lambda(f^p_\varepsilon(I))$$

We give the proof in Appendix 4.2.1.

Lemma 4.2.2. There are some $F > 0$ and some $\alpha > 0$ such that for every ε in $]0, \varepsilon_d[$, every interval I which satisfies the following properties:

(1) $\lambda(I) \leqslant \eta_0$

(2) $I \subset CJ$

and every interval K such that $f_\varepsilon^p(K) = I$ for some $p \in \mathbb{N}$ we have

$$\lambda(K) \leqslant F 2^{-\alpha s} \lambda(I)$$

where $s = \operatorname{card}\{j,\ 0 \leqslant j < p, f_\varepsilon^j(K) \cap J \neq \varnothing$ and $f_\varepsilon^{j+1}(K) \cap J = \varnothing\} + \operatorname{card}\{j,\ 0 \leqslant j \leqslant p, f_\varepsilon^j(K) \cap J = \varnothing\}$.

We give the proof in Appendix 4.2.2.

For every x in $[0, 1]$, let $\Sigma_{n,x} = \{y, f_\varepsilon^n(y) = x\}$ be the set of order n pre-images of x. Then $\operatorname{card} \Sigma_{n,x} = 2^n$. We define a map $\varphi_{n,x}$ from $\Sigma_{n,x}$ into $\{0, 1\}^n$ by $\varphi_{n,x}(y) = (i_0, \dots, i_{n-1})$ if and only if $i_q = [f_\varepsilon^q(y) + 1/2]$ for every q $0 \leqslant q < n$, $\varphi_{n,x}$ is obviously one-to-one and onto. If $h_{n,\varepsilon}(x)$ is the density of the nth iterate of the Lebesgue measure, we have from the expression of the Perron–Frobenius operator

$$h_{n,\varepsilon}(x) = \sum_{z \in \Sigma_{n,x}} \frac{1}{(f_\varepsilon^n)^{(1)}(z)}$$

Then we have the following:

Lemma 4.2.3. There is some $K' > 0$ such that for every $n \in \mathbb{N}$, every ε in $]0, \varepsilon_d[$ and every x and y in CJ

$$h_{n,\varepsilon}(x) \leqslant e^{K'(x-y)}\, h_{n,\varepsilon}(y)$$

Proof. (a) Assume that $|x - y| < \eta_0$. Let $z \in \Sigma_{n,x}$ and $\tilde{z} \in \Sigma_{n,y}$ be such that $\varphi_{n,x}(z) = \varphi_{n,y}(\tilde{z})$. Assuming $x < y$, we set $I = [x, y]$, $K = [z, \tilde{z}]$, and

$$M = \sup_{0 < \varepsilon < \varepsilon_d} \left\{ \frac{\sup_{x \in S^1} |f_\varepsilon^{(2)}(x)|}{\inf_{x \in S^1} f_\varepsilon^{(1)}(x)} \right\}$$

Then

$$\left| \log \left\{ \frac{(f_\varepsilon^n)^{(1)}(z)}{(f_\varepsilon^n)^{(1)}(\tilde{z})} \right\} \right| \leqslant M \sum_{j=0}^{n-1} |f_\varepsilon^j(z) - f_\varepsilon^j(y)|$$

We define two sequences $\{j_l,\ 1 \leqslant l \leqslant s\}$ and $\{j'_l,\ 1 \leqslant l \leqslant s\}$ as follows: For every l, $1 \leqslant l \leqslant s$,

$$f_\varepsilon^{j_l}(K) \cap J \neq \varnothing \text{ and } f_\varepsilon^{j_l - 1}(K) \cap J = \varnothing$$
$$f_\varepsilon^{j'_l}(K) \cap J \neq \varnothing \text{ and } f_\varepsilon^{j'_l + 1}(K) \cap J = \varnothing$$
$$f_\varepsilon^{t}(K) \cap J = \varnothing \text{ for every } t,\ j'_l < t < j_{l+1}$$

Then $0 \leqslant j_1 < j'_1 < \cdots < j_s < j'_s$, so that

$$\left| \log \left\{ \frac{(f_\varepsilon^n)^{(1)}(z)}{(f_\varepsilon^n)^{(1)}(\tilde{z})} \right\} \right| \leqslant M \left\{ \sum_{j=0}^{j_1 - 1} \lambda(f_\varepsilon^j(K)) + \sum_{q=1}^{s} \sum_{j=j_q}^{j'_q} \lambda(f_\varepsilon^j(K)) + \sum_{q=1}^{s-1} \sum_{j=j'_q+1}^{j_{q+1}-1} \lambda(f_\varepsilon^j(K)) + \sum_{j=j'_s+1}^{n-1} \lambda(f_\varepsilon^j)) \right\}$$

As $|x - y| \leqslant \eta_0$, we deduce from Lemma 4.2.2

$$\sum_{j=0}^{j_1 - 1} \lambda(f_\varepsilon^j(K)) + \sum_{q=1}^{s-1} \sum_{j=j'_q+1}^{j_{q+1}-1} \lambda(f_\varepsilon^j(K)) + \sum_{j=j'_s+1}^{n-1} \lambda(f_\varepsilon^j(K)) < F\,|x-y| \sum_{t=0}^{\infty} 2^{-\alpha t}$$

From Lemma 4.2.1 we have for every q, $1 \leqslant q \leqslant s$

$$\sum_{j=j_q}^{j'_q} \lambda(f_\varepsilon^j(K)) \leqslant C\lambda(f_\varepsilon^{j'_q + 1}(K))$$

Setting

$$s_q = \operatorname{card}\{j,\ 0 \leqslant j \leqslant j'_q,\ f_\varepsilon^j(K) \cap J \neq \varnothing \text{ and } f_\varepsilon^{j+1}(K) \cap J = \varnothing\} + \operatorname{card}\{j,\ 0 \leqslant j \leqslant j'_q + 1,\ f_\varepsilon^j(K) \cap J = \varnothing\}$$

we have then from Lemma 4.2.2.

$$\lambda(f_\varepsilon^{j'_q + 1}(K)) \leqslant |x-y|\, F e^{-\alpha s_q}$$

Therefore

$$\left| \log \left\{ \frac{(f_\varepsilon^n)^{(1)}(z)}{(f_\varepsilon^n)^{(1)}(z)} \right\} \right| \leqslant \frac{MF\,|x-y|}{1 - 2^{-\alpha}} + MC\,|x-y| \sum_{q=1}^{s} e^{-\alpha s_q}$$

and

$$\frac{(f_\varepsilon^n)^{(1)}(z)}{(f_\varepsilon^n)^{(1)}(z)} \leqslant e^{K'|x-y|}, \qquad \text{where } K' = \frac{MF(1+C)}{1 - 2^{-\alpha}}$$

Therefore,

$$\sum_{z\in\Sigma_{n,x}} \frac{1}{(f_\varepsilon^n)^{(1)}(z)} \leqslant e^{K'|x-y|} \sum_{\tilde{z}\in\Sigma_{n,y}} \frac{1}{(f_\varepsilon^n)^{(1)}(\tilde{z})}$$

and we have $h_{n,\varepsilon}(x) \leqslant e^{K'|x-y|} h_{n,\varepsilon}(y)$ when $|x-y| \leqslant \eta_0$.

(b) Assume now that $|x-y| > \eta_0$. Always assuming $y > x$, we introduce the sequence $\{x_i, i \in \{0,\dots [(y-x)/\eta_0] - 2\}\}$ defined by

$$x_0 = x$$

$$x_{i+1} = x_i + \eta_0 \qquad \text{for every } i,\ 0 \leqslant i \leqslant \left[\frac{y-x}{\eta_0}\right] - 2$$

$$x_{[(y-x)/\eta_0]} = y$$

we have then

$$\frac{h_{n,\varepsilon}(x)}{h_{n,\varepsilon}(y)} = \prod_{i=0}^{[(y-x)/\eta_0]-1} \frac{h_{n,\varepsilon}(x_i)}{h_{n,\varepsilon}(x_{i+1})} \leqslant e^{K'|y-x|} \qquad ∎$$

Lemma 4.2.4. $\rho_{N,\varepsilon}(x) = (1/N) \sum_{n=0}^{N-1} h_{n,\varepsilon}(x)$ converges uniformly on S^1 to the density $\rho_\varepsilon(x)$ of the invariant measure μ_ε when $N \to \infty$.

Proof. We give the proof in Appendix 4.2.4.

Theorem 4.2.1 is an obvious consequence of Lemmas 4.2.3 and 4.2.4.

4.2.2. Behavior of the Density ρ_ε when ε Goes to Zero. In this section we prove the following:

Theorem 4.2.2. Let $\mathscr{K}$ be any compact set such that $x_0^* \notin \mathscr{K}$ then ρ_ε converges uniformly to zero on $\mathscr{K}$ when $\varepsilon \to 0$.

That theorem shows that when $\varepsilon \to 0$ the invariant measure μ_ε converges to the Dirac measure $\delta_{x_0^*}$ for the weak topology on the space of measures. Everywhere in this section ε is positive.

Proposition 4.2.5. There is some $K' > 0$ such that for ε_d small enough and for every ε in $]0, \varepsilon_d[$, $\sup_{x\in\mathscr{K}} \rho_\varepsilon(x) < e^{K'} \cdot \inf_{x\in\mathscr{K}} \rho_\varepsilon(x)$

Proof. It is an obvious consequence of Theorem 4.2.1.

Lemma 4.2.6. There is some $\delta > 0$ such that for every ε_d in $]0, \varepsilon_d[$ $\inf_{x\in\mathscr{K}} \rho_\varepsilon(x) < (1/\delta)(a\varepsilon)^{1/2}$.

We give the proof in Appendix 4.2.6.

Lemma 4.2.7. There is some $\xi > 0$ such that

$$\sup_{x\in\mathscr{K}} \rho_\varepsilon(x) < \xi\varepsilon^{1/2} \qquad \text{for every } \varepsilon \in\ ]0, \varepsilon_d[$$

This is an obvious consequence of Proposition 4.2.5 and Lemma 4.2.6. Theorem 4.2.2 is an immediate consequence of that lemma. Theorem 4.2.2

implies that the expectation $\mu_\varepsilon(g)$ of any C^0 function g on S^1 converges to $g(x_0^*)$ when ε goes to zero with positive values. Since $\mu_\varepsilon(g)$ is a continuous function of ε_g on $]0, \varepsilon_d[$ (Theorem 4.1) and $]\varepsilon_c, 0[$ (then $\mu_\varepsilon(g) = g(x_\varepsilon^*)$) it is continuous on $]\varepsilon_c, \varepsilon_d[$. Therefore the invariant measure is weakly continuous on $]\varepsilon_c, \varepsilon_d[$ (although a bifurcation occurs at $\varepsilon = 0$).

4.3. Continuity of the Kolmogoroff–Sinai Entropy $h(\mu_\varepsilon)$

In this section we prove the following:

Theorem 4.3. The K–S entropy $h(\mu_\varepsilon)$ of the Bowen–Ruelle measure μ_ε is a continuous function of ε on $]\varepsilon_c, \varepsilon_d[$.

Lemma 4.3.1. $h(\mu_\varepsilon)$ is a C^0 function of ε on $[0, \varepsilon_d[$.

Proof. For any given ε_0 and ε in $]0, \varepsilon_d[$ we have

$$|\mu_\varepsilon(\log f_\varepsilon^{(1)}) - \mu_{\varepsilon_0}(\log f_{\varepsilon_0}^{(1)})| \leqslant \|\log f_\varepsilon^{(1)} - \log f_{\varepsilon_0}^{(1)}\|_{C^0} + |\mu_\varepsilon(\log f_{\varepsilon_0}^{(1)}) - \mu_{\varepsilon_0}(\log f_{\varepsilon_0}^{(1)})|$$

Since $\log f_\varepsilon^{(1)}$ and $\mu_\varepsilon(\log f_{\varepsilon_0}^{(1)})$ are continuous functions of ε (see Theorem 4.2.1), $\mu_\varepsilon(\log f_\varepsilon^{(1)})$ is also a C^0 function of ε. As $h(\mu_\varepsilon) = \mu_\varepsilon(\log f_\varepsilon^{(1)})$[13] the K–S entropy is continuous on $]0, \varepsilon_d[$. ∎

Lemma 4.3.2. $\lim_{\varepsilon\to 0,\,\varepsilon>0} h(\mu_\varepsilon) = 0$.

Proof. We have $h(\mu_\varepsilon) = \int_{CJ} (\log f_\varepsilon^{(1)}(x))\, \rho_\varepsilon(x)\, dx + \int_J (\log f_\varepsilon^{(1)}(x))\, \rho_\varepsilon(x)\, dx$ where $J =]x_0^* - \gamma,\ x_0^* + \gamma[$. We set $\alpha = \sup\{\|\log f_\varepsilon^{(1)}\|_{C^0},\ \varepsilon \in]0, \varepsilon_d[\}$. Then $h(\mu_\varepsilon) < \alpha\mu_\varepsilon(CJ) + \int_J (\log f_\varepsilon^{(1)}(x))\, \rho_\varepsilon(x)\, dx$. From Lemma 4.2.6 we have $\mu_\varepsilon(CJ) < \xi\sqrt{\varepsilon}$. As $f_\varepsilon^{(1)}(x) = (1 + a\varepsilon)/(1 - a(x - x_0^*))^2$ near x_0^* we have then

$$h(\mu_\varepsilon) < \alpha\xi\sqrt{\varepsilon} + \int_J \log(1 + a\varepsilon)\, \rho_\varepsilon(x)\, dx$$

$$+ 2 \int_J |\log(1 - a(x - x_0^*))|\, \rho_\varepsilon(x)\, dx$$

Hence for γ small enough

$$h(\mu_\varepsilon) < 2\alpha\xi\sqrt{\varepsilon} + 2a\gamma + 2a\varepsilon$$

We have then for every $\gamma > 0$ small enough $\lim_{\varepsilon\to 0,\,\varepsilon>0} h(\mu_\varepsilon) < 2a\gamma$. Therefore $\lim_{\varepsilon\to 0,\varepsilon>0} h(\mu_\varepsilon) = 0$. ∎

For $\varepsilon \in]\varepsilon_c, 0[$ the Bowen–Ruelle measure is the Dirac measure $\delta_{x_\varepsilon^*}$, so that $h(\mu_\varepsilon) = 0$ on $]\varepsilon_c, 0[$. Therefore $h(\mu_\varepsilon)$ is continuous at $\varepsilon = 0$. (From Lemma 4.3.2) which proves the continuity of $h(\mu_\varepsilon)$ on $]\varepsilon_c, \varepsilon_d[$.

5. CONCLUSION

We now give a brief physical interpretation of those result. Imagine a physical experiment explained by type-I intermittency. Then the measured mean value of any physical quantity will continuously evolve with the control parameter. Besides, as soon as the threshold is crossed, the system will be stochastic but as the rate of stochasticity (measured by $h(\mu_\varepsilon)$) increases up from zero with the control parameter it will take a very long time to observe any sensitive dependence on initial conditions.

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APPENDIX 2.3: PROOF OF THE HYPERBOLICITY (THEOREM 2.3)

We set $L_\varepsilon = \sup\{q \in \mathbb{N},\ \forall p \in \{0,..., q\}\ f_\varepsilon^p(x_{1,\varepsilon}) \leqslant x_{3,\varepsilon}\}$ and $I_\varepsilon = \sup\{q \in \mathbb{N},\ \forall p \in \{0,..., q\},\ f_\varepsilon^p(x_{1,\varepsilon}) \leqslant x_{4,\varepsilon}\}$, where $x_{4,\varepsilon} = x_0^* + (1/a)[1 - (1 + a\varepsilon)^{1/2}]$ satisfies $f_\varepsilon^{(1)}(x_{4,\varepsilon}) = 1$. $f_\varepsilon^{(1)}$ is increasing and smaller than 2 outside $[x_{1,\varepsilon}, x_{3,\varepsilon}]$. Taking into account the nature of the regularization, we have then

$$\inf_{S^1} (f_\varepsilon^{I_\varepsilon})^{(1)}(x) \geqslant (f_\varepsilon^{I_\varepsilon})^{(1)}(x_{1,\varepsilon})$$

$f_\varepsilon^{(1)}$ is increasing and smaller than 1 on $[x_{1,\varepsilon}, x_{4,\varepsilon}]$; it is greater than 1 outside this region. Therefore $\inf_{S^1}(f_\varepsilon^q)^{(1)}(x) \geqslant (f_\varepsilon^{I_\varepsilon})^{(1)}(x_{1,\varepsilon})$ for every q such that $q < L_\varepsilon$. Hence, for every $n \in \mathbb{N}$, we have

$$(f_\varepsilon^n)^{(1)}(x) \geqslant ((f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon}))^{[n/L_\varepsilon]} (f_\varepsilon^{I_\varepsilon})^{(1)}(x_{1,\varepsilon})$$

Proposition 2.3.1. For ε_d small enough and $ax_0^* < 2/3$ there is some number $\alpha >^* 1$ such that for any ε in $]0, \varepsilon_d[$,

$$(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon}) > \alpha$$

Proof. From the definition of I_ε, we have

$$f_\varepsilon^{L_\varepsilon}(x_{1,\varepsilon}) > \varphi_\varepsilon(x_{3,\varepsilon})$$

From Proposition 2.2.1 we have then

$$(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon}) > \frac{\varepsilon + a(\varphi_\varepsilon(x_{3,\varepsilon}) - x_0^*)^2}{\varepsilon + a(x_{1,\varepsilon} - x_0^*)^2}$$

Hence for ε_d small enough and some $\alpha > 1$,

$$(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon}) \geqslant \alpha > 1 \quad \blacksquare$$

From Proposition 2.3.1 we deduce

$$\inf_{S^1}(f_\varepsilon^n)^{(1)}(x) > \{(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon})\}^{n/L_\varepsilon} F(\varepsilon) \qquad \text{for } \varepsilon \in \,]0, \varepsilon_d[$$

where $F(\varepsilon) = (f_\varepsilon^{I_\varepsilon})^{(1)}(x_{1,\varepsilon})/(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon})$ is a continuous positive nonvanishing function.

Proposition 2.3.2. For ε_d small enough, there is some number $d > 1$ such that $\{(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon})\}^{1/L_\varepsilon} > d^{\sqrt{\varepsilon}}$ for any ε in $]0, \varepsilon_d[$.

Proof. We first estimate L_ε. From Proposition 2.2.2 we have

$$L_\varepsilon = \frac{\arctan[(a/\varepsilon)^{1/2}(x_{3,\varepsilon} - x_0^*)] - \arctan[(a/\varepsilon)^{1/2}(x_{1,\varepsilon} - x_0^*)]}{\arctan(a\varepsilon)^{1/2}}$$

So that for ε_d small enough $L_\varepsilon < 2\pi/a\varepsilon^{1/2}$ for ε in $]0, \varepsilon_d[$. From Proposition 2.3.1 we have $\log((f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon})) > \log \alpha$. Therefore, setting $d = \alpha^{\sqrt{a}/2\pi} > 1$, we have

$$\{(f_\varepsilon^{L_\varepsilon})^{(1)}(x_{1,\varepsilon})\}^{1/L_\varepsilon} > d^{\sqrt{\varepsilon}} \qquad \blacksquare$$

From Propositions 2.3.1 and 2.3.2 we deduce

$$\inf_{S^1}(f_\varepsilon^n)^{(1)}(x) > F(\varepsilon) d^{n\sqrt{\varepsilon}}$$

which ends the proof of Theorem 2.3.

APPENDIX 2.4.2: CONTINUITY OF THE LEBESGUE MEASURE OF FUNDAMENTAL INTERVALS (PROOF OF LEMMA 2.4.2)

Let us denote by $\mathscr{P}_n$ the set of order n pre-images of 1/2. This set is in a one-to-one correspondence with $\{0, 1\}^n$ since we can associate to any point x in $\mathscr{P}_n$ an itinerary $(i_1,..., i_n) \in \{0, 1\}^n$ defined by

$$i_{n-k} = [f_\varepsilon^k(x) + \tfrac{1}{2}]$$

For every $p \in \mathbb{N}$ and every itinerary $i \in \{0, 1\}^p$ we define $x_{p,i}(\varepsilon)$ as the location of the corresponding preimage of 1/2 and we prove that those preimages continuously depend on ε.

Proposition 2.4.3. There is some positive C^0 function $\chi(\varepsilon)$ on $]0, \varepsilon_d[$ such that for any ε in $]0, \varepsilon_d[$, any $p \in \mathbb{N}$ and any $i \in \{0, 1\}^p$ the following inequality holds on a neighborhood of ε:

$$|x_{p,i}(\varepsilon') - x_{p,i}(\varepsilon)| < \chi(\varepsilon)\,|\varepsilon' - \varepsilon|$$

Proof. For given p and i, we define for every k, $0 \leqslant k \leqslant p$

$$x_k(\varepsilon') = f_{\varepsilon'}^{p-k}(x_{p,i}(\varepsilon'))$$

We introduce the sequence $\{\xi_{\varepsilon',k}(x_{k-1}(\varepsilon')),\ k \in \{1,\dots, p\}\}$ defined as follows:

$$\text{if } x_{k,\varepsilon} \in [0, 1/2], \qquad \xi_{\varepsilon',k}(x_{k-1}(\varepsilon')) = \varphi_{\varepsilon'}(x_k(\varepsilon'))$$
$$\text{if } x_{k,\varepsilon'} \in [1/2, 1], \qquad \xi_{\varepsilon',k}(x_{k-1}(\varepsilon')) = \psi_{\varepsilon'}(x_k(\varepsilon'))$$

As φ_ε is a C^1 function of ε and $\partial\psi_\varepsilon(x)/\partial\varepsilon = 0$, we obtain by an easy induction:

$$\frac{d}{d\varepsilon'} x_{p,i}(\varepsilon')\bigg|_{\varepsilon'=\varepsilon} = \sum_{k=1}^{p-1} \left\{ \left[\frac{\partial}{\partial\varepsilon'} \xi_{\varepsilon',k}(x_{k-1}(\varepsilon'))\bigg|_{\varepsilon'=\varepsilon} \right] \prod_{j=k+1}^{p} \xi_{\varepsilon,j}^{(1)}(x_{j-1}(\varepsilon')) \right\}$$
$$+ \frac{\partial}{\partial\varepsilon'} \xi_{\varepsilon',p}(x_{p-1}(\varepsilon'))\bigg|_{\varepsilon'=\varepsilon}$$

From Theorem 2.3, f_ε is expanding. Hence

$$\prod_{j=k+1}^{p} \xi_{\varepsilon,j}^{(1)}(x_{j-1}(\varepsilon')) < \frac{1}{F(\varepsilon)}\, d^{-(p-k)\sqrt{\varepsilon}}$$

and, as $F(\varepsilon) < 1$ on $]0, \varepsilon_d[$ we have

$$\left| \frac{d}{d\varepsilon'} x_{p,i}(\varepsilon')\bigg|_{\varepsilon'=\varepsilon} \right| < \frac{1}{F(\varepsilon)} \sum_{k=1}^{p} d^{-(p-k)\sqrt{\varepsilon}} \left| \frac{\partial}{\partial\varepsilon'} \xi_{\varepsilon',k}(x_{k-1,\varepsilon'})\bigg|_{\varepsilon'=\varepsilon} \right|$$

We set $B(\varepsilon) = \sup_{S^1}(\partial/\partial\varepsilon')\,\varphi_{\varepsilon'}(x)|_{\varepsilon'=\varepsilon})$. $B(\varepsilon)$ is a C^0 function on $]0, \varepsilon_d[$ and

$$\left| \frac{d}{d\varepsilon'} x_{p,i}(\varepsilon')\bigg|_{\varepsilon'=\varepsilon} \right| < \frac{B(\varepsilon)}{F(\varepsilon)} \frac{1}{1 - d^{-\sqrt{\varepsilon}}}$$

Setting $\chi(\varepsilon) = [4B(\varepsilon)/F(\varepsilon)][1/(1 - d^{-\sqrt{\varepsilon}})]$, we have then

$$\left| \frac{d}{d\varepsilon'} x_{p,i}(\varepsilon')\bigg|_{\varepsilon'=\varepsilon} \right| < \frac{\chi(\varepsilon)}{4}$$

Hence, on a neighborhood of ε

$$|x_{p,i}(\varepsilon') - x_{p,i}(\varepsilon)| < \frac{\chi(\varepsilon)}{2} |\varepsilon' - \varepsilon| \qquad \blacksquare$$

Now we prove Lemma 2.4.2.

Any order p fundamental interval has its end points in $\{0\} \cup \{1\} \cup \mathscr{P}_p \cup \mathscr{P}_{p-1}$. Hence, from Proposition 2.4.3, we have for ε' in a neighborhood of any given ε in $]0, \varepsilon_d[$

$$|\lambda(I_{p,i,\varepsilon'}) - \lambda(I_{p,i,\varepsilon})| < \chi(\varepsilon) |\varepsilon' - \varepsilon| \qquad \blacksquare$$

APPENDIX 2.5.1: PROOF OF LEMMA 2.5.1

As f_ε is a C^3 endomorphism and $f_\varepsilon^{(1)}$ and $f_\varepsilon^{(2)}$ are, respectively, C^2 and C^1 functions of the parameter ε, we deduce from the Taylor formula:

$$f_\varepsilon(x) = x + \varepsilon + a(x - x_0^*)^2 + r(\varepsilon, x)$$

with

$$r(\varepsilon, x) = \varepsilon(x - x_0^*) \left[\frac{\partial}{\partial \varepsilon} f_\varepsilon^{(1)}(x_0^*) \Big|_{\varepsilon=0} + \frac{\varepsilon}{2} \frac{\partial}{\partial \varepsilon^2} f_\varepsilon^{(1)}(x_0^*) \Big|_{\varepsilon=\varepsilon_1} + \frac{(x - x_0^*)}{2} \frac{\partial}{\partial \varepsilon} f_\varepsilon^{(1)}(x_0^*) \Big|_{\varepsilon=\varepsilon_2} + \frac{(x - x_0^*)^3}{6} f_\varepsilon^{(3)}(\xi) \right]$$

where $0 < \varepsilon_1 < \varepsilon$, $0 < \varepsilon_2 < \varepsilon$ and ξ lies between x_0^* and x. For ε_d small enough and some $\beta > 0$, there is some $K > 0$ such that $|r(\varepsilon, x)| < K |x - x_0^*| (\varepsilon + (x - x_0^*)^2)$ for $0 < \varepsilon < \varepsilon_d$ and $|x - x_0^*| < \beta$. For any y in $]x_0^* - \beta/2, x_0^* + \beta/2[$ the equation $y = f_\varepsilon(x)$ has only one solution $x \in]\varphi_\varepsilon(x_0^* - \beta/2), \varphi_\varepsilon(x_0^* + \beta/2)[$ according to the implicit function theorem. Moreover, for ε_d and β small enough:

(1) $x \in]x_0^* - \beta, x_0^* + \beta[$

(2) $x = y - \varepsilon - a(y - x_0)^2 + \tilde{r}(\varepsilon, y)$

(3) $|\tilde{r}(\varepsilon, y)| \leqslant K |y - x_0^*| (\varepsilon + (y - x_0^*)^2)$

For given x_0 and n (satisfying the assumptions of Lemma 2.5.1.1) we define the sequence $\{t_p, p \in \{0,..., n\}\}$ by $t_p = x - x_0^*$.

Proposition 2.5.2. If ε_d is small enough $0 < t_p < \beta/2$ for every p $0 \leqslant p \leqslant n$.

Proof. The proof is recursive and left to the reader.

Then we have for every p in $\{0,\dots,n\}$, $t_{p+1} = t_p - \varepsilon a t_p^2 + \tilde{r}(\varepsilon, t_p + x_0^*)$. For $(\theta, v) \in \mathbb{Z}^2$ we define a sequence $\{v_p, p \in \{0,\dots,n\}\}$ as follows: $v_0 = t_0$, $v_1 = t_1$, and for $p \geqslant 1$, $v_{p+1} = v_p + a v_p^2 - v(\beta/p)^3 - \theta K(v_p^3 + v_p(\beta/p)^3)$. Let $\{y_p, p \in \{0,\dots,n\}\}$ be the sequence defined by

$$y_p = v_p - \frac{1}{a(p+\tau)}, \qquad \text{where} \quad \tau = \frac{1}{a t_0}$$

Then we have the following:

Proposition 2.5.5:

(1) $y_0 = 0$;

(2) for β and ε_d small enough $|y_1| \leqslant 1/(\tau+1)^2$;

(3) for $p \in \{1,\dots,n-1\}$

$$y_{p+1} = y_p \left(1 - \frac{2}{p+\tau}\right) - a y_p^2 - v\left(\frac{\beta}{p}\right)^3 - \theta K \left[y_p + \frac{1}{a(p+\tau)}\right]^3$$
$$- \theta K \left(\frac{\beta}{p}\right)^3 \left[y_p + \frac{1}{a(p+\tau)}\right] - \frac{1}{a(p+\tau)^2(p+\tau+1)}$$

Proof. The proof is left to the reader.

Let us define $\{\sigma_p, p \in \{0,\dots,n\}\}$, by $\sigma_p = |y_p|\,(p+\tau)^2$. Then we have the following:

Proposition 2.5.6. Assuming β small enough, $\sigma_p \leqslant (p+\tau)^{1/2}$ for every p in $\{0,\dots,n\}$.

Proof. The proof is recursive and left to the reader. From that proposition we deduce

$$y_p \leqslant \frac{\sigma_p}{(p+\tau)^2} \leqslant \frac{1}{(p+\tau)^{3/2}}$$

so that

$$\left| v_p - \frac{1}{a(p+\tau)} \right| \leqslant \frac{1}{(p+\tau)^{3/2}}$$

We now reach the end of the proof. From $t_{p+1} = t_p - \varepsilon - a t_p^2 + \tilde{r}(\varepsilon, t_p + x_0^*)$ and $|\tilde{r}(\varepsilon, y)| \leqslant K\,|y - x_0^*|\,(\varepsilon + (y - x_0^*)^2)$ we deduce for $p \geqslant 1$

$$t_p - a t_p^2 + K\left\{t_p^3 + \frac{\beta^3 t_p}{p^3}\right\} \geqslant t_{p+1} \geqslant t_p - a t_p^2 - K\left\{t_p^3 + \frac{t_p \beta^3}{p^3}\right\}$$

if we denote by $\{v_p^+\}$ the sequence $\{v_p\}$ obtained for $(\theta, \nu) = (1, 0)$ and $\{v_p^-\}$ the sequence obtained for $(\theta, \nu) = (1, 1)$ we have therefore

$$v_p^+ \geqslant t_p \geqslant v_p^- \text{ for every } p, \qquad 0 \leqslant p \leqslant n$$

Hence, applying Proposition 2.5.6 to the sequences $\{\sigma_p^+\}$ and $\{\sigma_p^-\}$ associated with $\{v_p^+\}$ and $\{v_p^-\}$ we obtain

$$\left| t_p - \frac{1}{a(p+\tau)} \right| \leqslant \frac{1}{(p+\tau)^{3/2}} \text{ for every } p, \qquad 0 \leqslant p \leqslant n$$

which proves Lemma 2.5.1(1).

The proof of Lemma 2.5.1(2), which is quite similar, is left to the reader.

APPENDIX 3.2.1: PRELIMINARY RESULT

For any x_0 in $[\varphi_0(x_0^* + \beta/(\log \beta)^2),\ x_0^* + \beta/(\log \beta)^2]$ we define the sequence $\{x_n,\ n \in \mathbb{N}\}$ by $x_{n+1} = \varphi_0(x_n)$. And for any $\tilde{x}_0$ in $[x_0, x_0^* + \beta]$ we define $\{\tilde{x}_n, n \in \mathbb{N}\}$ by $\tilde{x}_{n+1} = \varphi_0(\tilde{x}_n)$. Then we have the following:

Lemma 3.2.3. For β small enough there is some $N \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $n > N$:

$$|\tilde{x}_n - x_n| \leqslant \frac{2}{a\beta(\log \beta)^2} \frac{1}{(n+1)^{3/2}}$$

Proof. Since $x_0 \geqslant \varphi_0(x_0^* + \beta/(\log \beta)^2)$ we have $x_0 \geqslant x_0^* + \beta/2(\log \beta)^2$ for β small enough, so that if we set $\tau = 1/a(x_0 - x_0^*)$, $1 < \tau \leqslant 2(\log \beta)^2/a\beta$. From Lemma 2.5.2 we have

$$|\tilde{x}_n - x_n| \leqslant \frac{1}{(n+\tau)^{3/2}} + \frac{1}{(n+\tau')^{3/2}} + \frac{1}{a} \left| \frac{1}{n+\tau} - \frac{1}{n+\tau'} \right|$$

where $\tau' = 1/a(x_0' - x_0^*)$. Hence

$$|\tilde{x}_n - x_n| \leqslant \frac{2}{(n+1)^{3/2}} + \frac{\tau}{a(n+1)^2}$$

so that

$$|\tilde{x}_n - x_n| \leqslant \frac{2}{a\beta(\log \beta)^2 (n+1)^{3/2}} \left[a\beta(\log \beta)^2 + \frac{2(\log \beta)^2}{a(n+1)^{1/2}} \right]$$

For β small enough, $a\beta(\log\beta)^2 < 1/2$ and if we choose $N > (16/a^2)(\log\beta)^8$ we have for $n > N$

$$|\tilde{x}_n - x_n| \leqslant \frac{2}{a\beta(\log\beta)^2}\frac{1}{(n+1)^{3/2}} \qquad \blacksquare$$

Proof of Lemma 3.2.1. (a) First, we assume that for some $p \leqslant n$

$$f_0^p(K_n) \cap \left[x_0^*, x_0^* + \frac{\beta}{(\log\beta)^2}\right] \neq \varnothing$$

Let

$$k_0 = \sup\left\{q \in \{0,..., n\}, f_0^q(K_n) \cap \left[x_0^*, x_0^* + \frac{\beta}{(\log\beta)^2}\right] \neq \varnothing\right\}$$

and

$$k_1 = \sup\{q \in \{0,..., n\}, f_0^q(K_n) \cap [x_0^*, x_0^* + \beta] \neq \varnothing\}$$

We assume that $k_0 > N$ (N is defined in Lemma 3.2.2).

Proposition 3.2.4. For β small enough $k_1 - k_0 \geqslant (\log\beta)^2/4a\beta$.

Proof. We set $l = \inf\{x, x \in f_0^{k_1}(K_n)\}$. From the definition of k_1 we have $f_0(l) > x_0^* + \beta$ so that for β small enough $x_0^* + \beta > l > x_0^* + \beta - 2a\beta^2 > x_0^* + \beta/2$. From the definition of k_0 we have

$$\varphi_0^{k_1-k_0}(l) \leqslant x_0^* + \frac{\beta}{(\log\beta)^2}$$

From Lemma 2.5.2, we have

$$\left|\varphi_0^{k_1-k_0}(l) - x_0^* - \frac{1}{a(k_1-k_0+\tau)}\right| \leqslant \frac{1}{(k_1-k_0+\tau)^{3/2}}$$

where $\tau = 1/a(l - x_0^*)$.

Therefore

$$\frac{1}{a(k_1-k_0+\tau)} - \frac{1}{(k_1-k_0+\tau)^{3/2}} < \frac{\beta}{(\log\beta)^2}$$

so that for β small enough $k_1 - k_0 + \tau \geqslant (\log\beta)^2/2a\beta$ and, as $\tau < 2/a\beta$ (since $l > x_0^* + \beta/2$), we have for β small enough $k_1 - k_0 \geqslant (\log\beta)^2/4a\beta$. $\blacksquare$

Proposition 3.2.5. $f_0^{k_0}(K_n) \subset [x_0^*, x_0^* + \beta]$.

Proof. We set

$$r_0 = \sup_{x \in f_0^{k_0}(K_n)} x \quad \text{and} \quad r_1 = \sup_{x \in f_0^{k_1}(K_n)} x$$

For every j in $\{1,..., k_1 - k_0\}$, $f_0^{k_0+j}(r_0) > x_0^* + \beta$. Indeed, assume that for some j in $\{1,..., k_1 - k_0\}$

$$f_0^{k_0+j}(r_0) \leqslant x_0^* + \beta$$

Then, since, from the definition of k_0,

$$\sup_{x \in f_0^{k_0+j}(K_n)} x > x_0^* + \frac{\beta}{(\log \beta)^2}$$

we have $\lambda(f_0^{k_0+j}(K_n)) < \beta - \beta/(\log \beta)^2$. On the other hand, since $\lambda(f_0^{k_0}(K_n)) > \beta - \beta/(\log \beta)^2$ and $f_0^{(1)}(x) \geqslant 1$ on $[x_0^*, 1]$ we have $\lambda(f_0^{k_0+j}(K_n)) > \beta - \beta/(\log \beta)^2$. A contradiction. Assume now that $f_0^{k_0}(K_n)$ is not contained in $[x_0^*, x_0^* + \beta]$. Then $r_0 > x_0^* + \beta$. Since $f_0^{(1)}(x) \geqslant 1/(1 - a\beta)^2$ on $[x_0^* + \beta, 1]$ and $f_0^{k_0+j}(r_0) > x_0^* + \beta$ for every j $0 \leqslant j \leqslant k_1 - k_0$, we have $r_0 - x_0^* \leqslant (1 - a\beta)^{2(k_1 - k_0)}$.

From Proposition 3.2.4 we have then $r_0 - x_0^* \leqslant (1 - a\beta)^{(\log \beta)^2/2a\beta}$ so that for β small enough $r_0 - x_0^* \leqslant \beta$ thus contradicting the initial assumption. Therefore $f_0^{k_0}(K_n) \subset [x_0^*, x_0^* + \beta]$. ∎

Now we consider the sets $f_0^l(K_n)$, $k_0 < l \leqslant n$.

Proposition 3.2.6. There is some $\rho > 1$ (which does not depend on n) such that for $n \geqslant l > k_0$

$$\lambda(f_0^l(K_n)) \leqslant \rho^{l-n}$$

Proof. For $n \geqslant l > k_0$, $f_0^l(K_n) \cap [x_0^*, x_0^* + \beta/(\log \beta)^2] = \varnothing$ from the definition of k_0. If we set $\rho = f_0^{(1)}(x_0^* + \beta/(\log \beta)^2) > 1$ we have $f_0^{(1)}(x) \geqslant \rho$ on $f_0^l(K_n)$ for every l, $k_0 < l \leqslant n$. Therefore $\lambda(f_0^l(K_n)) \leqslant \rho^{l-n} \lambda(f_0^n(K_n)) \leqslant \rho^{l-n}$. ∎

We now consider the sets $f_0^l(K_n)$, $0 \leqslant l \leqslant k_0$. We define for $0 \leqslant l \leqslant k_0$ a sequence $\{K^l, l \in \{0,..., k_0\}$ of subsets as follows:

$$K^{k_0} = f_0^{k_0}(K_n)$$
$$k^{l-1} = f_0(K^l)$$

Then we have the following:

Proposition 3.2.7. For every l in $\{0,\dots,k_0\}$, either

$$K^l = f_0^l(K_n) \qquad \text{or} \qquad \sup_{x\in K^l} x \leqslant \inf_{y\in f_0^l(K_n)} y$$

Proof. The proof is recursive, using the monotonicity of φ_0 and ψ_0; it is left to the reader.

Now we compare the Lebesgue measures of $f_0^l(K_n)$ and K^l.

Proposition 3.2.8. For every l, $0 \leqslant l \leqslant k_0$

$$\lambda(f_0^l(K_n)) \leqslant \lambda(K^l)$$

Proof. Let l be any integer in $\{0,\dots,k_0-1\}$. If $K^l = f_0^l(K_n)$ then $K^{l+1} = f_0(K^l) = f_0^{l+1}(K_n)$ so that

$$\frac{\lambda(K^l)}{\lambda(K^{l+1})} = \frac{\lambda(f_0^l(K_n))}{\lambda(f_0^{l+1}(K_n))}$$

if not, then $\sup_{x\in K^l} x \leqslant \inf_{x\in f_0^l(K_n)} x$. Since $K^{k_0} = f_0^{k_0}(k_n) \subset [x_0^*, x_0^* + \beta]$, we have also

$$K^l \subset [x_0^*, x_0^* + \beta]$$

Then, we use Lemma 2.5.3 and we get $\sup_{x\in K^l} f_0^{(1)}(x) \leqslant \inf_{x\in f_0^l(K_n)} f_0^{(1)}(x)$. As $\lambda(K^{l+1}) \leqslant \sup_{x\in K^l} f_0^{(1)}(x)\cdot\lambda(K^l)$ and $\lambda(f_0^{l+1}(K_n)) \geqslant \inf_{x\in f_0^l(K_n)} f_0^{(1)}(x)\cdot \lambda(f_0^l(K_n)$ we have then

$$\frac{\lambda(K^l)}{\lambda(K^{l+1})} \geqslant \frac{\lambda(f_0^l(K_n))}{\lambda(f_0^{l+1}(K_n))}$$

so that, as $f_0^{k_0}(K_n) = K^{k_0}$, we have for every l, $0 \leqslant l \leqslant k_0$,

$$\lambda(f_0^l(K_n)) \leqslant \lambda(K^l) \qquad \blacksquare$$

Now we majorize $\lambda(K^l)$.

Proposition 3.2.9. For every l, $0 \leqslant l < k_0 - N$ (N is defined in Lemma 3.2.1),

$$\lambda(K^l) \leqslant \frac{2}{a\beta(\log\beta)^2}\,\frac{1}{(k_0-l+1)^{3/2}}$$

Proof. (a) We set $l_0 = \inf_{x\in K^{k_0}} x$. From the definition of k_0, we have $x_0^* < l_0 < x_0^* + \beta/(\log\beta)^2$. As $K^{k_0} \subset [x_0^*, x_0^* + \beta]$ (see Proposition 3.2.5) we have

$$x_0^* < l_0 < r_0 < x_0^* + \beta$$

Therefore, we can use Lemma 3.2.3 and we get

$$\lambda(K^l) \leqslant \frac{2}{a\beta(\log\beta)^2} \frac{1}{(k_0 - l + 1)^{3/2}} \text{ for every } l, \qquad 0 \leqslant l < k_0 - N \qquad \blacksquare$$

Now we prove Lemma 3.2.1 under the preceding assumptions:

(1) For some $k \in \mathbb{N}$, $\quad f_0^k(K_n) \cap \left[x_0^*, x_0^* + \frac{\beta}{(\log\beta)^2}\right] \neq \varnothing$

(2) $k_0 > N$

As in the proof of Lemma 3.1.3, we have for (a, b) in K_n^2

$$\left|\log\left\{\frac{(f_0^n)^{(1)}(a)}{(f_0^n)^{(1)}(b)}\right\}\right| \leqslant L_0 \sum_{p=0}^{n-1} |f_0^p(a) - f_0^p(b)|,$$

$$\text{where} \quad L_0 = \sup_{x \in [x_0^*, 1]} |f_0^{(2)}(x)|$$

we have $\sum_{k_0-N}^{k_0} \lambda(f_0^p(K_n)) \leqslant N + 1$. We deduce from Proposition 3.2.8 and Proposition 3.2.9

$$\sum_{p=0}^{k_0-N-1} \lambda(f_0^p(K_n)) \leqslant \frac{2}{a\beta(\log\beta)^2} \sum_{j=0}^{\infty} \frac{1}{j^{3/2}} \leqslant \frac{4}{a\beta(\log\beta)^2}$$

and from Proposition 3.2.6 we have

$$\sum_{p=k_0}^{N-1} \lambda(f_0^p(K_n)) \leqslant \sum_{k=0}^{\infty} \frac{1}{\rho^k} \leqslant \frac{\rho}{\rho - 1}$$

Therefore

$$\left|\log\left\{\frac{(f_0^n)^{(1)}(a)}{(f_0^n)^{(1)}(b)}\right\}\right| \leqslant L_0 \left\{N + 1 + \frac{\rho}{\rho - 1} + \frac{4}{a\beta(\log\beta)^2}\right\}$$

(b) Assume now that there is some $k \in \mathbb{N}$ such that $f_0^l(K_n) \cap [x_0^*, x_0^* + \beta/(\log\beta)^2] \neq \varnothing$ but $k_0 \leqslant N$. Then we no longer need Proposition 3.2.9 and we have

$$\left|\log\left\{\frac{(f_0^n)^{(1)}(a)}{(f_0^n)^{(1)}(b)}\right\}\right| \leqslant L_0 \left\{\sum_{p=0}^{k_0} \lambda(f_0^p(K_n)) + \sum_{p=k_0+1}^{n-1} \lambda(f_0^p(K_n))\right\}$$

$$\leqslant L_0 \left\{N + 1 + \frac{\rho}{\rho - 1}\right\}$$

(c) Assume that for every p, $0 \leqslant p \leqslant n$, $f_0^p(K_n) \cap [x_0^*, x_0^* + \beta/(\log\beta)^2] = \varnothing$. We set $\rho = f_0^{(1)}(x_0^* + \beta/(\log\beta)^2) > 1$. For every p, $0 \leqslant p \leqslant n$, $\inf_{x \in f_0^p(K_n)} f_0^{(1)}(x) > \rho$. Then by the same proof we already used in Lemma 3.1.3 we have

$$\left| \log \left\{ \frac{(f_0^n)^{(1)}(a)}{(f_0^n)^{(1)}(b)} \right\} \right| \leqslant \frac{L_0\rho}{\rho - 1}$$

(d) We set

$$A = \sup \left\{ \frac{L_0\rho}{\rho - 1}, L_0 \left(N + 1 + \frac{\rho}{\rho - 1} + \frac{4}{a\beta(\log\beta)^2} \right) \right\} \quad \text{and } \rho = \exp A$$

We have then for every $n \in \mathbb{N}$ and every component of E_n except I_n

$$\frac{1}{\delta} \leqslant \frac{(f_0^n)^{(1)}(a)}{(f_0^n)^{(1)}(b)} \leqslant \delta \qquad \blacksquare$$

APPENDIX 4.1.5: PROOF OF LEMMA 4.1.5

We define centered k cylinders as follows: for every $k \in \mathbb{N}$ for every $(i_{-k},..., i_0,..., i_k) \in \{0, 1\}^{2k+1}$, $V^k(i_{-k},..., i_k) = \{\bar{j} \in \{0, 1\}^{\mathbb{Z}}, j_p = i_p$ for every p, $-k \leqslant p \leqslant k\}$. Then we have the following:

Lemma 4.1.6. The measure $\tilde{\lambda}_\varepsilon(V^p_{i_{-p},\dots,i_p})$ of any given centered p cylinder is a continuous function of ε on $]0, \varepsilon_d[$. That result was proved by Dobrushin[12] for potentials such as ours. To every C^0 function g on $[0, 1]$ we associate a one parameter family of C^0 functions on $\{0, 1\}^{\mathbb{Z}}$ $\{\tilde{g}_\varepsilon, 0 < \varepsilon < \varepsilon_d\}$ thus defined: $\tilde{g}_\varepsilon(\bar{i}) = \bar{g}_\varepsilon(\tau(\bar{i}))$ where τ is the projection from $\{0, 1\}^{\mathbb{Z}}$ onto $\{0, 1\}^{\mathbb{N}}$. We define on $\mathscr{C}^0(\{0, 1\}^{\mathbb{Z}}, \mathbb{R})$ the following C^0 norm: $\|\tilde{g}\|_{C^0} = \sup_{\bar{i} \in \{0,1\}^{\mathbb{Z}}} |\tilde{g}(\bar{i})|$.

We call a k-cylinder function any C^0 function which takes constant values on centered k cylinders of $\{0, 1\}^{\mathbb{Z}}$. Then we have the following:

Lemma 4.1.7. There is some sequence $\{\tilde{g}^n_\varepsilon, n \in \mathbb{N}\}$ which converges to $\tilde{g}_\varepsilon$ in $\mathscr{C}^0(\{0, 1\}^{\mathbb{Z}}, \mathbb{R})$, $\| \ \|_{C^0})$ such that for every n, $\tilde{g}^n_\varepsilon$ is an n-cylinder function.

Proof. The existence follows from the Stone–Weierstrass theorem.

Lemma 4.1.8. For every ε_0 in $]0, \varepsilon_d[$, $\tilde{\lambda}_\varepsilon(\tilde{g}_{\varepsilon_0})$ is a continuous function of ε at ε_0.

This is a obvious consequence of Lemma 4.1.6 and Lemma 4.1.7.

We now prove Lemma 4.1.5.

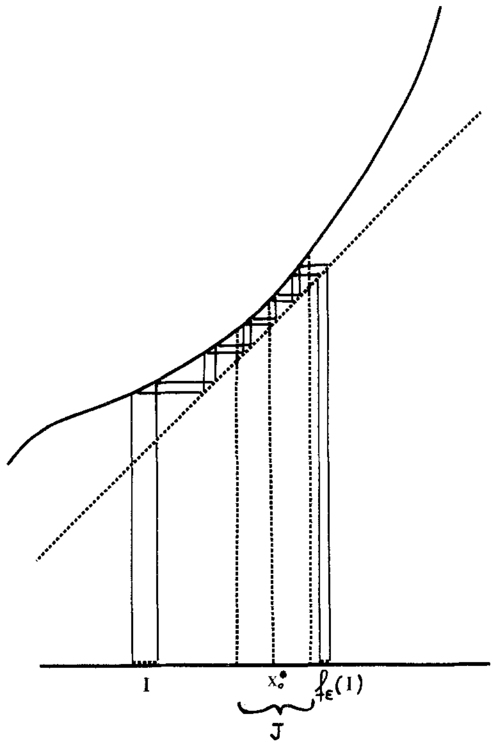

Fig. 7. Successive images of an interval I in the region around x_0^*.

We have $|\overline{\mu_\varepsilon}(\overline{g_\varepsilon}) - (\overline{\mu_{\varepsilon_0}}(\overline{g_{\varepsilon_0}})| < |\overline{\mu_\varepsilon}(\overline{g_{\varepsilon_0}}) - \overline{\mu_{\varepsilon_0}}(\overline{g_{\varepsilon_0}})| + \| \overline{g_\varepsilon} - \overline{g_{\varepsilon_0}} \|_{C^0}$ and from Lemma 4.1.3 and Lemma 4.1.8 we deduce that $\overline{\mu_\varepsilon}(\overline{g_\varepsilon})$ is a continuous function of ε.

APPENDIX 4.2.1. PRELIMINARY RESULTS

(1) First we prove that there are some $\zeta > 0$ and some $\zeta' > 0$ such that $\zeta > p\sqrt{\varepsilon} > \zeta$.

Since $\lambda(I) \leqslant \eta_0$, $I \cap J = \varnothing$ and $f_\varepsilon(I) \cap J \neq \varnothing$, and from the definition of η_0, we have

$$I \subset]\varphi_\varepsilon^2(x_0^* - \gamma), x_0^* - \gamma[$$

Let l be the smallest integer such that $f_\varepsilon^l(x_0^* - \gamma) > x_0^* + \gamma$. From Proposition 7.2.2 we have $l > [2/(a\varepsilon)^{1/2}] \arctan[\gamma(a/\varepsilon_d)^{1/2}]$. Therefore as $p \geqslant l$ we have $p\sqrt{\varepsilon} > \zeta$ where $\zeta = (2/\sqrt{a}) \arctan[\gamma(a/\varepsilon_d)^{1/2}]$ for ε_d small enough. Besides $p \leqslant l + 2 < 2\pi/(\varepsilon a)^{1/2}$ so that $p\sqrt{\varepsilon} < \zeta'$ where $\zeta' = 2\pi/\sqrt{a}$.

(2) Then we prove that $\lambda(I) < D\lambda(f_\varepsilon^p(I))$ for some $D > 0$.

We have $\lambda(f_\varepsilon^p(I)) = \lambda(I)(f_\varepsilon^p)^{(1)}(\xi)$ where $\xi \in]\varphi_\varepsilon(x_0^* - \gamma), x_0^* - \gamma[$. Therefore $\lambda(f_\varepsilon^p(I)) \geqslant \lambda(I)(f_\varepsilon^p)^{(1)}(\varphi_\varepsilon^2)^{(1)}(\varphi_\varepsilon^2(x_0^* - \gamma))$. Assuming that $\varepsilon_d < \gamma$ and $a\gamma < 1$ we deduce from Proposition 2.2.2 that for ε_d and γ small enough $\lambda(I) \leqslant D\lambda(f_\varepsilon^p(I))$ for some $D > 0$.

(3) Lemma 4.2.8. For β and ε_d small enough and $\eta_0 < \beta/2$ there is some $L > 0$ such that for every in $]0, \varepsilon_d[$ and any $n \leqslant \beta\varepsilon^{-1/3}$ we have the following:

(a) For any x_0 in $]x_0^* + \beta/4, x_0^* + \beta/2[$ and any $\tilde{x}_0$ in $]x_0, x_0 + \eta_0[$, the sequences $\{x_p, 0 \leqslant p \leqslant n\}$ and $\{\tilde{x}_p, 0 \leqslant p \leqslant n\}$ defined by $x_{p+1} = \varphi_\varepsilon(x_p)$ and $\tilde{x}_{p+1} = \varphi_\varepsilon(\tilde{x}_p)$ satisfy $|\tilde{x}_p - x_p| \leqslant (L/p^2)|\tilde{x}_0 - x_0|$ for every p, $1 \leqslant p \leqslant n$.

(b) For any $\tilde{x}_0$ in $]x_0^* - \beta/2, x_0^* - \beta/4[$ and any x_0 in $]\tilde{x}_0 - \eta_0, \tilde{x}_0[$ the sequences $\{x_p, 0 \leqslant p \leqslant n\}$ and $\{\tilde{x}_p, 0 \leqslant p \leqslant n\}$ defined by $x_{p+1} = f_\varepsilon(x_p)$ and $\tilde{x}_{p+1} = f_\varepsilon(\tilde{x}_p)$ satisfy $|\tilde{x}_p - x_p| \leqslant (L/p^2)|\tilde{x}_0 - x_0|$ for every p, $1 \leqslant p \leqslant n$.

Proof of Lemma 4.2.8. We prove the first part and leave the proof of the second part to the reader. As φ_ε is increasing an $[0, 1]$ and $\tilde{x}_0 > x_0$ we have for every p, $0 \leqslant p \leqslant n$, $x_p > \tilde{x}_p$. Besides there is some $\xi_p \in]x_{p+1}, \tilde{x}_{p+1}[$ such that $\tilde{x}_p - x_p = (\tilde{x}_{p+1} - x_{p+1}) f_\varepsilon^{(1)}(\xi_p)$. As $x_0^* < x_{p+1} < x_0 < x_0^* + \beta/2$, $\tilde{x}_{p+1} < \tilde{x}_0 < x_0 + \eta_0 < x_0^* + \beta$ and $f_\varepsilon^{(1)}$ is increasing on $[x_0^*, x_0 + \beta]$ we have $f_\varepsilon^{(1)}(\xi_p) > f_\varepsilon^{(1)}(x_{p+1})$. Hence, setting $S_p = 1/\prod_{j=1}^p f_\varepsilon^{(1)}(x_j)$ we have $\tilde{x}_p - x_p \leqslant S_p(\tilde{x}_0 - x_0)$ for every p, $1 \leqslant p \leqslant n$.

Now we majorize S_p.

Proposition 4.2.9. For β and ε_d small enough there is some $L > 0$ such that $S_p < L/p^2$ for every p, $1 \leqslant p \leqslant n$.

Proof. For every x in $]x_0^*, x_0^* + \beta[$ we have

$$f_\varepsilon^{(1)}(x) = 1 + 2a(x - x_0^*) + S(\varepsilon, x) \qquad \text{where}$$

$$S(\varepsilon, x) = \varepsilon \left(\frac{\partial}{\partial\varepsilon} f_\varepsilon^{(2)}(x_0^*)\right)\bigg|_{\varepsilon=\varepsilon_2} (x - x_0{}^*) + \varepsilon \left(\frac{\partial}{\partial\varepsilon} f_\varepsilon^{(1)}(x_0^*)\right)\bigg|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\left(\frac{\partial^2}{\partial\varepsilon^2} f^{(1)}(x_0^*)\right)\bigg|_{\varepsilon=\varepsilon_1} + \frac{(x - x_0^*)^2}{2} f_\varepsilon^{(3)}(\xi)$$

and $\xi \in]x_0^*, x[$, $0 < \varepsilon_1 < \varepsilon$, $0 < \varepsilon_2 < \varepsilon$. Therefore we have for some $K > 0$:

$$f_\varepsilon^{(1)}(x_j) \geqslant 1 + 2a(x_j - x_0^*) - K(\varepsilon + (x_j - x_0)^2)$$

for every j, $1 \leqslant j \leqslant n$.

From Lemma 2.5.1 we have $|x_j^* - x_0 - 1/a(j+\tau)| \leqslant 1/(j+\tau)^{3/2}$ where $\tau = 1/a(x_0 - x_0^*)$. Therefore for $\varepsilon_d < 1/K$ we have

$$f_\varepsilon^{(1)}(x_j) \geqslant (1 - K\varepsilon)\left[1 - \frac{\delta}{(j+\tau)^{3/2}}\right]\left(1 + \frac{2}{j+\tau}\right) \qquad \text{where}$$

$$\delta = \frac{a}{1 - K\varepsilon_d} \frac{1 + K(a+1)^2}{a^3}$$

so that for β and ε_d small enough,

$$S_p \leqslant \frac{1}{N} \exp\left(-2 \sum_{j=1}^{p} \frac{1}{j}\right) \exp\left[-2 \sum_{j=1}^{p} \left(\frac{1}{j+\tau} - \frac{1}{j}\right)\right]$$

where $N = \exp(-2\beta K \varepsilon_d^{2/3}) \exp[-2\delta\zeta(3/2)] \exp[-4\zeta(2)]$ and ζ denotes the Riemann zeta function.

There is some $B > 0$ such that $|\log p - \sum_{j=1}^{p} 1/j| < B$. Hence $S_p \leqslant L/p^2$ for any $L > (1/N) \exp(2B) \exp[(16/a\beta)\, \zeta(2)]$.

Lemma 4.2.8 follows from Proposition 4.2.9.

Proof of Lemma 4.2.1. We assume in what follows $3\gamma < \beta < 4\gamma$. Since $p > \xi\varepsilon^{-1/2}$ we have for ε_d small enough $p > 2([\beta\varepsilon^{-1/3}] + 1)$. Hence we can write

$$\sum_{l=1}^{p-1} \lambda(f_\varepsilon^l(I)) = \sum_{l=1}^{[\beta\varepsilon^{-1/3}]} \lambda(f_\varepsilon^l(I)) + \sum_{l=[\beta\varepsilon^{-1/3}]+1}^{p-[\beta\varepsilon^{-1/3}]-1} \lambda(f_\varepsilon^l(I)) + \sum_{l=p-[\beta\varepsilon^{-1/3}]}^{p-1} \lambda(f_\varepsilon^l(I))$$

Since $\lambda(I) \leqslant \eta_0$ and $\lambda(f_\varepsilon^p(I)) < \eta_0$, we deduce from Lemma 4.2.8 that

$$\sum_{l=1}^{p-1} \lambda(f_\varepsilon^l(I)) \leqslant [\lambda(I) + \lambda(f_\varepsilon^p(I))] \frac{\pi^2 L}{6} + \Sigma_\varepsilon$$

where we denote by Σ_ε the expression

$$\sum_{l=[\beta\varepsilon^{-1/3}]+1}^{p-[\beta\varepsilon^{-1/3}]-1} \lambda(f_\varepsilon^l(I))$$

Now we majorize Σ_ε. Assuming I is a closed interval, we set $f_\varepsilon^l(I) = [x_l, \tilde{x}_l]$. Then we deduce from Lemma 2.5.1

$$|x_{l_0} - x_0^*| \leqslant \frac{1}{(l_0+\tau)^{3/2}} + \frac{1}{a(l_0+\tau)}, \qquad \text{where } l_0 = [\beta\varepsilon^{-1/3}] + 1$$

$$\text{and} \qquad \tau = \frac{1}{a(x_0^* - x_0)}$$

Hence, for ε_d small enough $|x_0^* - x_{l_0}| < (2/a\beta)\varepsilon^{1/3}$. We have also $|x_0^* - x_{p-l_0}| < (2/a\beta)\varepsilon^{1/3}$ (the proof is left to the reader). Therefore as f_ε is increasing, we have for every q, $l_0 \leqslant q \leqslant p - l_0$, $|x_0^* - x_q| < (2/a\beta)\varepsilon^{1/3}$.

On $]x_0^* - \beta, x_0^* + \beta[$ we have $f_\varepsilon(x) = x + \varepsilon + a(x - x_0^*)^2 + r(\varepsilon, x)$ where $|r(\varepsilon, x)| < K'\{|x - x_0^*|^3 + \varepsilon|x - x_0^*|\}$ with some $K' > 0$ which does not depend on β. Then we have the following:

Proposition 4.2.10. For $\beta < 1/2K'$ and every point x which satisfies for some $n \in \mathbb{N}$ $|f_\varepsilon^j(x) - x_0^*| < \beta$ for every j, $0 \leqslant j < n$, we have

$$f_\varepsilon^j(x) \geqslant x + \frac{j\varepsilon}{2} \text{ for every } j, \qquad 0 \leqslant j \leqslant n$$

Proof. The proof is recursive. It is left to the reader

Proposition 4.2.11. There is some $q \in \mathbb{N}$ such that $f_\varepsilon^q(x_{l_0}) > x_0^* + (2/a\beta)\varepsilon^{1/3}$. The proof is left to the reader.

We set $l_1 = \inf\{q \in \mathbb{N}, f_\varepsilon^q(x_{l_0}) > x_0^* + (2/a\beta)\varepsilon^{1/3}\}$. Then we have the following:

Proposition 4.2.12. $\Sigma_\varepsilon \leqslant \sum_{q=l_0+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I))$.

Proof. We have $|f_\varepsilon^{p-l_0}(x_0 - x_0^*)| < (2/a\beta)\varepsilon^{1/3}$ and $f_\varepsilon^{l_0+l_1}(x_0) > x_0^* + (2/a\beta)\varepsilon^{1/3}$. As f_ε is increasing, we have therefore $p - l_0 < l_0 + l_1$ so that

$$\Sigma_\varepsilon \leqslant \sum_{q=l_0+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I)) \qquad \blacksquare$$

We assume $\varepsilon_d < \beta^{3/2}$.

Proposition 4.2.13. If ε_d is chosen small enough, we have for every ε in $]0, \varepsilon_d[$

(1) $f_\varepsilon^{(1)}(x) > 1$ on $]x_0^* + \varepsilon^{2/3}, x_0^* + \beta[$

(2) $f_\varepsilon^{(1)}(x) < 1$ on $]x_0^* - \beta, x_0^* - \varepsilon^{2/3}[$

(3) $|f_\varepsilon^{(1)}(x) - 1| \leqslant 8a\,\varepsilon^{2/3}$ on $[x_0^* - 2\varepsilon^{2/3}, x_0^* + 2\varepsilon^{2/3}]$

The proof is left to the reader.

We recall that $f_\varepsilon^q(I) = [x_q, \tilde{x}_q]$. We set $l_2 = \sup\{q \in \mathbb{N}, x_q < x_0^* + \varepsilon^{2/3}\}$ and $l_3 = \sup\{q \in \mathbb{N}, \tilde{x}_q < x_0^* - \varepsilon^{2/3}\}$. We have for ε_d small enough $l_0 < l_3 < l_2 < l_0 + l_1$ so that

$$\Sigma_\varepsilon \leqslant \sum_{q=l_0+1}^{l_3} \lambda(f_\varepsilon^q(I)) + \sum_{q=l_3+1}^{l_2} \lambda(f_\varepsilon^q(I)) + \sum_{l_2+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I))$$

Proposition 4.2.14. For ε_d small enough, we have for every ε in $]0, \varepsilon_d[$

(1) $\sum_{q=l_0+1}^{l_3} \lambda(f_\varepsilon^q(I)) \leqslant \frac{16L}{a\beta^3}\lambda(I)$ (L is defined in Lemma 4.2.8)

(2) $\sum_{q=l_2+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I)) \leqslant \frac{32\pi L}{\sqrt{a}\,\beta^{3/4}}\lambda(f_\varepsilon^p(I))$

Proof. (1) For every q, $l_0 < q \leqslant l_3$, $f_\varepsilon^q(I) \subset]x_0^* - \beta, x_0^* - \varepsilon^{2/3}[$. Therefore $\sum_{q=l_0+1}^{l_3} \lambda(f_\varepsilon^q(I)) \leqslant (l_3 - l_0)\,\lambda(f_\varepsilon^{l_0}(I))$. From Lemma 4.2.5 we deduce $\lambda(f_\varepsilon^{l_0}(I)) < (1/l_0^2)\,\lambda(I)$. Hence

$$\sum_{q=l_0+1}^{l_3} \lambda(f_\varepsilon^q(I)) \leqslant \frac{l_3 - l_0}{l_0^2}\lambda(I)$$

From Proposition 4.2.10 we have

$$x_0^* - \varepsilon^{2/3} > x_{l_3} > x_{l_0} + \frac{(l_3 - l_0)\varepsilon}{2} > x_0^* - \frac{2\varepsilon^{1/3}}{a\beta} + (l_3 - l_0)\frac{\varepsilon}{2}$$

so that, for ε_d small enough $l_3 - l_0 < 4\varepsilon^{-2/3}/a\beta$. And as $l_0 > \beta\varepsilon^{-1/3}/2$ we have then

$$\sum_{q=l_0+1}^{l_3} \lambda(f_\varepsilon^q(I)) \leqslant \frac{16}{a\beta^3}L\lambda(I)$$

(2) For every q, $l_2 < q \leqslant l_0 + l_1$, $f_\varepsilon^q(I) \subset]x_0^* + \varepsilon^{2/3}, x_0^* + \beta[$. From Proposition 4.2.13 we have then $\lambda(f_\varepsilon^{q+1}(I)) \geqslant \lambda(f_\varepsilon^q(I))$ for every q, $l_2 < q \leqslant l_0 + l_1$ so that

$$\sum_{q=l_2+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I)) \leqslant (l_0 + l_1 - l_2)\,\lambda(f_\varepsilon^{l_0+l_1}(I)) \leqslant (l_0 + l_1)\,\lambda(f_\varepsilon^{l_0+l_1}(I))$$

As $l_0 + l_1 > p - l_0$ we deduce from Lemma 4.2.8

$$\lambda(f_\varepsilon^{l_0+l_1}(I)) < \frac{L}{(p - l_0 - l_1)^2}\lambda(f_\varepsilon^p(I))$$

so that

$$\sum_{l_2+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I)) \leqslant \frac{l_0 + l_1}{(p - l_0 - l_1)^2}L\lambda(f_\varepsilon^p(I))$$

One easily shows, using Proposition 2.2.2, that for ε_d small enough

$$l_0 + l_1 < \frac{2\pi}{\sqrt{a}}\varepsilon^{-1/2} \qquad \text{and} \qquad p - (l_0 + l_1) > \frac{\beta}{4}\varepsilon^{-1/3}$$

Therefore

$$\sum_{q=l_2+1}^{l_0+l_1} \lambda(f_\varepsilon^q(I)) \leqslant \frac{32\pi}{\sqrt{a}\,\beta^{3/4}} L\lambda(f_\varepsilon^p(I))$$

since $\varepsilon_d < \beta^{3/2}$. ∎

Proposition 4.2.15. For ε_d small enough we have

$$\sum_{q=l_3+1}^{l_2} \lambda(f_\varepsilon^q(I)) \leqslant 128L\lambda(f_\varepsilon^p(I)) \qquad \text{for every } \varepsilon \text{ in }]0, \varepsilon_d[$$

Proof. For ε_d small enough, we have

$$f_\varepsilon^q(I) \subset [x_0^* - 2\varepsilon^{2/3}, x_0^* + 2\varepsilon^{2/3}] \text{ for every } q,\ l_3 < q \leqslant l_2$$

(the proof is left to the reader).

From Proposition 4.2.13 we have then

$$\sum_{q=l_3+1}^{l_2} \lambda(f_\varepsilon^q(I)) \leqslant \frac{\lambda(f_\varepsilon^{l_2}(I))}{8a\varepsilon^{2/3}} [(1 + 8a\varepsilon^{2/3})^{l_3-l_2} - 1]$$

From Proposition 2.2.2 we deduce $l_2 - l_3 < 4\varepsilon^{-1/3}$ for ε_d small enough.

We have $f_\varepsilon^{(1)}(xl_2) > 1$ (the proof is left to the reader). Therefore $f_\varepsilon^{(1)}(x) > 1$ on $f_\varepsilon^q(I)$ for every q, $l_2 \leqslant q < l_0 + l_1$ so that

$$\sum_{l_3+1}^{l_2} \lambda(f_\varepsilon^q(I)) \leqslant 8\varepsilon^{-1/3} \lambda(f_\varepsilon^{l_0+l_1}(I)) \leqslant 128L\lambda(f_\varepsilon^p(I)) \qquad ∎$$

Using Propositions 4.2.14 and 4.2.15 and the second of the preliminary results we prove Lemma 4.2.1:

$$\sum_{l=1}^{p-1} \lambda(f_\varepsilon^l(I)) \leqslant C\lambda(f_\varepsilon^p(I)) \qquad \text{for } \varepsilon_d \text{ small enough}$$

where

$$C = \frac{\pi^2 L}{6}(1 + D) + 128L\left(1 + \frac{D}{6a\beta^3} + \frac{\pi}{4\sqrt{a}\,\beta^{3/4}}\right) \qquad ∎$$

APPENDIX 4.2: PROOF OF LEMMA 4.2.2

We set $C = [x_{1,\varepsilon}, x_0^* - \gamma]$, $T = [x_0^* + \gamma, x_{2,\varepsilon}]$, and $E = [0, x_{1,\varepsilon}[\cup]x_{2,\varepsilon}, 1]$. We have $\lambda(I) = (f_\varepsilon^p)^{(1)}(\xi)\,\lambda(K)$ for some $\xi \in K$. Hence, setting $\xi_j = f_\varepsilon^j(\xi)$ for every j, $0 \leqslant j < p$, we have $\lambda(I) = \lambda(K) \prod_{j=0}^{p-1} f_\varepsilon^{(1)}(\xi_j)$.

Proposition 4.2.16. Let x be any point such that for some $n \in \mathbb{N}$:

(1) $f_\varepsilon^n(x) \in E$

(2) $f_\varepsilon^j(x) \in CE$ for every j, $0 \leqslant j < n$

Then

$$\prod_{j=0}^{n-1} f_\varepsilon^{(1)}(x)) > 1$$

Proof. This is an obvious consequence of Proposition 2.2.2.

We now define two ordered sequences of integers $\{t_q, 1 \leqslant q \leqslant N\}$ and $\{t'_q, 1 \leqslant q \leqslant N'\}$ as follows:

(1) For every q in $\{1,\dots, N\}$, $0 \leqslant t_q < p$, $\xi_{t_q - 1} \in E$ and $\xi_{t_q} \in CE$ (except of course if $t_q = 0$).

(2) For every q in $\{1,\dots, N'\}$, $0 \leqslant t'_q < p$, $\xi_{t'_q - 1} \in CE$ and $\xi_{t'_q} \in E$ (except if $t'_q = 0$).

(3) Those sequences are maximal.

Then $|N - N'| \leqslant 1$ and if we set $A = \{j, 0 \leqslant j < p, \xi_j \in E\}$.

Proposition 4.2.17:

(1) If $I \subset E$, $(f_\varepsilon^p)^{(1)}(\xi) > 2^{\operatorname{card} A}$

(2) If $I \cap CE \neq \varnothing$, $(f_\varepsilon^p)^{(1)}(\xi) > 2^{\operatorname{card} A}(f_\varepsilon^{p-t_N})^{(1)}(\xi_{t_N})$.

Proof. (1) Assume $N = N'$. Then we have

$$0 \leqslant t_1 < t'_1 < \cdots < t_N < t'_N < p$$

so that

$$(f_\varepsilon^p)^{(1)}(\xi) = (f_\varepsilon^{t_1})^{(1)}(\xi) \prod_{q=1}^{N} (f_\varepsilon^{t'_q - t_q})(\xi_{t_q}) \prod_{q=1}^{N-1} (f_\varepsilon^{t_{q+1} - t'_q})^{(1)}(\xi_{t'_q})$$
$$\times (f_\varepsilon^{p - t'_N})^{(1)}(\xi_{t'_N})$$

From Proposition 4.2.16 we have then

$$(f_\varepsilon^p)^{(1)}(\xi) > (f_\varepsilon^{t_1})^{(1)}(\xi)(f_\varepsilon^{p-t'_N})^{(1)}(\xi_{t'_N}) \prod_{q=1}^{N-1} (f_\varepsilon^{t_{q+1} - t'_q})^{(1)}(\xi_{t'_q})$$

Since $f_\varepsilon^{(1)}(x) = 2$ on the subset E and $\operatorname{card} A = p - t'_N + t_1 + \sum_{q=1}^{N-1} t_{q+1} - t'_q$, we have then $(f_\varepsilon^p)^{(1)}(\xi) > 2^{\operatorname{card} A}$. If $N' = N + 1$, the proof is similar.

(2) The proof is similar and left to the reader.

For every $x \in C$, we define $\xi_\varepsilon^c(x) = \sup\{q \in \mathbb{N},\ \forall j \in \{0,..., q\}, f_\varepsilon^j(x) \in C\}$. Similarly for every x in T we define $\xi_\varepsilon^T(x) = \sup\{q \in \mathbb{N}, \forall j \in \{0,..., q\}\, f_\varepsilon^j(x) \in T\}$.

Proposition 4.2.18. There is some $0 < G < 1$ such that for every ε in $]0, \varepsilon_d[$ and any intervals I and K satisfying the assumptions of Lemma 4.2.2

$$\frac{\operatorname{card} A}{s - \operatorname{card} A} > G$$

Proof. From Proposition 2.2.2 there is some $P \in \mathbb{N}$ such that for every x in C, every y in T and every ε in $]0, \varepsilon_d[$

$$\xi_\varepsilon^c(x) < P \qquad \text{and} \qquad \xi_\varepsilon^T(y) < P$$

Therefore the number of successive iterates in any connected component of $CJ - E$ is majorized and since any orbit leaving the subset T must visit region E before entering region C or region T, at least N' points of the orbit belong to E and at most $2NP$ to $CJ - E$. We have then $\operatorname{card} A > N'$ and $s - \operatorname{card} A < 2NP$.

Therefore, setting $G = 1/4P$ (assuming $N > 1$) we have

$$\frac{\operatorname{card} A}{s - \operatorname{card} A} > G \qquad \blacksquare$$

From Proposition 4.2.17 and 2.2.2 we have, setting

$$\delta = \frac{\gamma^2}{(x_0^* - x_{1,0})^2}, \quad (f_\varepsilon^p)^{(1)}(\xi) > \delta 2^{\operatorname{card} A}$$

From Proposition 4.2.18 we have $\operatorname{card} A > [G/(1 - G)]^s$. Therefore setting $\alpha = G/(1 - G)$ and $F = 1/\delta$ we have

$$\lambda(K) \leqslant \lambda(I)\, F 2^{-\alpha s} \qquad \blacksquare$$

APPENDIX 4.2.4: PROOF OF LEMMA 4.2.4

Proposition 4.2.19. For every $n \in \mathbb{N}$, every ε in $]0, \varepsilon_d[$ and every x in CJ, we have

$$h_{n,\varepsilon}(x) < \frac{e^{K'}}{1-2\gamma}$$

Proof. This is an obvious consequent of Lemma 4.2.3.

Proposition 4.2.20. There is some $A > 0$ such that $h_{n,\varepsilon}(x)$ is Lipschitz on CJ with Lipschitz constant A for every $n \in \mathbb{N}$ and every ε in $]0, \varepsilon_d[$.

Proof. We have

$$|h_{n,\varepsilon}(x) - h_{n,\varepsilon}(y)| < h_{n,\varepsilon}(x) \left| 1 - \frac{h_{n,\varepsilon}(y)}{h_{n,\varepsilon}(x)} \right|$$

Hence, from Lemma 4.2.3 and Proposition 4.2.19 we have

$$|h_{n,\varepsilon}(x) - h_{n,\varepsilon}(y)| < \frac{e^{K'}}{1-2\gamma}(1 - e^{-K'|x-y|}) < A\,|x-y|$$

where $A = e^{K'}\alpha/(1-2\gamma)$ and $\alpha = \sup(2K', 2(1-e^{-K'}))$. ∎

From that proposition, we deduce that for every ε in $]0, \varepsilon_d[$ the sequence $\{h_{n,\varepsilon}(x), n \in \mathbb{N}\}$ is equicontinuous on CJ. Therefore $\{\rho_{n,\varepsilon}(x), n \in \mathbb{N}\}$ is also equicontinuous on CJ.

We know from Lasota and Yorke[14] that $\rho_{n,\varepsilon}(x)$ converges to $\rho_\varepsilon(x)$ in L^1. From that convergence, the continuity of the function ρ_ε and the equicontinuity of the sequence $\{\rho_{n,\varepsilon}, n \in \mathbb{N}\}$ on CJ, we deduce that $\rho_{n,\varepsilon}$ converges uniformly to ρ_ε on CJ.

APPENDIX 4.2.6: PROOF OF LEMMA 4.2.6

First we prove the following:

Proposition 4.2.21. For every $n \in \mathbb{N}$ and every $\varepsilon \in]0, \varepsilon_d[$ $\mu_\varepsilon[(\varphi_\varepsilon^{n+1}(1/2), \varphi_\varepsilon^n(1/2)]) = \mu_\varepsilon([1/2, \psi_\varepsilon \circ \varphi_\varepsilon^n(1/2)])$.

The proof is recursive and founded on the invariance of μ_ε. It is left to the reader.

Proposition 4.2.22. For ε_d small enough we have

$$\sup\{n, \varphi_\varepsilon^n(1/2) > x_{1,\varepsilon}\} > \left[\frac{1}{(a\varepsilon)^{1/2}}\right] \qquad \text{for every } \varepsilon \in \,]0, \varepsilon_d[$$

The proof is left to the reader.

Proposition 4.2.23. For every ε in $]0, \varepsilon_d[$ there is some $p(\varepsilon) \in \mathbb{N}$, such that

(1) $p(\varepsilon) < \left[\frac{1}{(a\varepsilon)^{1/2}}\right]$

(2) $\mu_\varepsilon([1/2, \psi_\varepsilon \circ \varphi_\varepsilon^{p(\varepsilon)}(1/2)]) < (a\varepsilon)^{1/2}$

Proof. We have

$$\sum_{n=0}^{[(1/a\varepsilon)^{1/2}]-1} \mu_\varepsilon([\varphi_\varepsilon^{n+1}(1/2), \varphi_\varepsilon^n(1/2)]) < 1$$

Besides for every $n \in \mathbb{N}$, $\mu_\varepsilon([\varphi_\varepsilon^{n+1}(1/2), \varphi_\varepsilon^n(1/2)]) > 0$. Therefore, there is some $p(\varepsilon) < [1/(a\varepsilon)^{1/2}]$ such that $\mu_\varepsilon([\varphi_\varepsilon^{p(\varepsilon)+1}(1/2), \varphi_\varepsilon^{p(\varepsilon)}(1/2)]) < (a\varepsilon)^{1/2}$. From Proposition 4.2.21 we have then $\mu_\varepsilon([1/2, \psi_\varepsilon \circ \varphi_\varepsilon^{p(\varepsilon)}(1/2)]) < (a\varepsilon)^{1/2}$. ∎

We set

$$\inf_{\varepsilon \in]0, \varepsilon_d[} (\psi_\varepsilon \circ \varphi_\varepsilon^{p(\varepsilon)}(1/2) - 1/2) = \delta$$

From Proposition 4.2.22 we easily deduce that $\delta > 0$. Then from Proposition 4.2.23 we have $\inf_{x \in CJ} \rho_\varepsilon(x) < (1/\delta\,(a\varepsilon)^{1/2})$, and from Lemma 4.2.5, we have

$$\sup_{x \in CJ} \rho_\varepsilon(x) < \xi\sqrt{\varepsilon} \qquad \text{where} \quad \xi = \frac{e^{K'}\sqrt{a}}{\delta}$$

Therefore we have for $\mathscr{K} \subset CJ$

$$\sup_{x \in \mathscr{K}} \rho_\varepsilon(x) < \xi\sqrt{\varepsilon}$$

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